# Implementation of algorithms and data structures

8. seminar

Jirka Fink

https://ktiml.mff.cuni.cz/~fink/

Department of Theoretical Computer Science and Mathematical Logic
Faculty of Mathematics and Physics
Charles University in Prague

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# Matching

## Definitions

- $(V, E)$ is an undirected graph on $n$ vertices and $m$ edges
- $M \subseteq E$ is a matching if every vertex is incident with at most one edge of $M$
- A vertex $v \in V$ is *$M$-covered* if some edge of $M$ is incident with $v$; otherwise $v$ is *$M$-exposed*
- Matching $M$ is perfect if all vertices are $M$-covered
- Matching is maximal if there is no matching having more edges

## Related problems

- Find a maximum matching
- Find a perfect matching
- Find minimum-weight perfect matching for a given weights of edges
- Find maximum-weight matching

Algorithms depends on whether a graph is bipartite or it contains an odd cycle

# Alternating cycles

## Definition

Let $M \subseteq E$ be a matching of a graph $G$

- A path $P$ is *$M$-alternating* if its edges are alternately in and not in $M$.
- An $M$-alternating path is *$M$-augmenting* if both end-vertices are $M$-exposed.

## Augmenting path theorem of matchings

A matching $M$ in a graph $G = (V, E)$ is maximum if and only if there is no $M$-augmenting path.

## Notes

- Algorithms for matching are based on finding augmenting paths
- However, testing non-existence of a alternating path is impractical

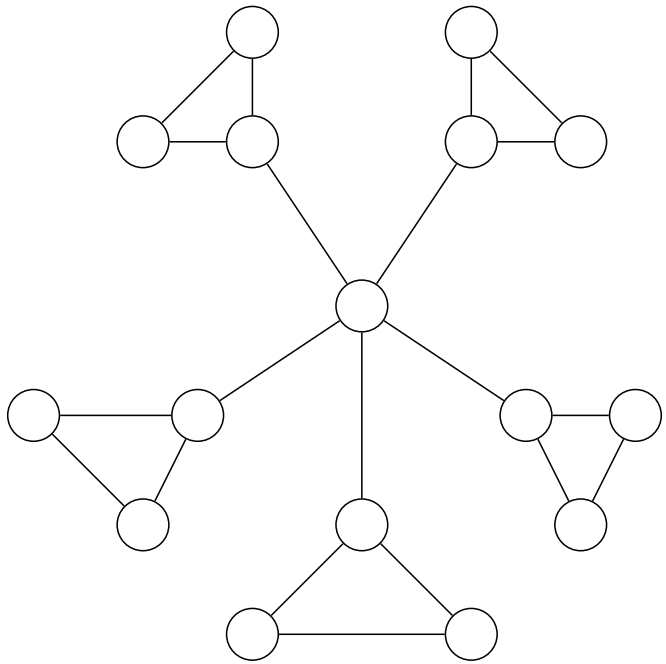
# Tutte-Berge Formula

## Definition

- Let $\mathrm{defic}(G) = |V| - 2|M|$ be the number of exposed vertices by a maximum size matching in $G$
- Let $oc(G)$ be the number of odd components of a graph $G$

## Observations

- $\mathrm{defic}(G) \geq \mathrm{oc}(G)$
- For every $A \subseteq V$ it holds that $\mathrm{defic}(G) \geq \mathrm{oc}(G \setminus A) - |A|$.



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## Tutte's matching theorem

A graph $G$ has a perfect matching if and only if $\mathrm{oc}(G \setminus A) \leq |A|$ for every $A \subseteq V$.

## Theorem: Tutte-Berge Formula

$\mathrm{defic}(G) = \max\{\mathrm{oc}(G \setminus A) - |A|;\ A \subseteq V\}$

# Building an alternating tree

## Initialization of $M$-alternating tree $T$ on vertices $A \dot\cup B$

$T = A = \emptyset$ and $B = \{r\}$ where $r$ is an $M$-exposed root. ①

## Use $uv \in E$ to extend $T$

Input: An edge $uv \in E$ such that $u \in B$ and $v \notin A \cup B$ and $v$ is $M$-covered.

Action: Let $vz \in M$ and extend $T$ by edges $\{uv, vz\}$ and $A$ by $v$ and $B$ by $z$.

## Properties

- $r$ is the only $M$-exposed vertex of $T$.
- Vertices in $A$ have only one child in $T$ which is connected by an edge of $M$
- For every $v$ of $T$, the path in $T$ from $v$ to $r$ is $M$-alternating.
- $|B| = |A| + 1$

## Use $uv \in E$ to augment $M$

Input: An edge $uv \in E$ such that $u \in B$ and $v \notin A \cup B$ and $v$ is $M$-exposed.

Action: Let $P$ be the path obtained by attaching $uv$ to the path from $r$ to $u$ in $T$. Replace $M$ by $M \triangle E(P)$.

1. An $M$-alternating tree $T$ with the root $r$ on vertices $A$ and $B$ is a tree obtained from this initialization by applying the following operation extend.

# Frustrated tree

## Definition

*M*-alternating tree *T* is *M*-frustrated if every edge of *G* having one end vertex in *B* has the other end vertex in *A*. ①

## Observation

If a bipartite graph *G* has a matching *M* and an frustrated *M*-alternating tree, then *G* has no perfect matching. ② ③

1. That is, an $M$-alternating tree is frustrated if neither operation extend nor augment can be applied. Note that in bipartite graphs, there is no edge between vertices of $B$.
2. $B$ are single vertex components in the graph $G \setminus A$. Therefore, $\mathrm{oc}(G \setminus A) \geq |B| > |A|$.
3. This proves that Tutte's matching theorem for bipartite graphs: From every $M$-exposed vertex $r$ we build an $M$-alternating tree $T$ such that $T$ can be used to augment $M$ to cover $r$ or $T$ is frustrated.

# Algorithm for perfect matching problem in a bipartite graph

## Algorithm

```
1  Init: M := ∅
2  while G contains an M-exposed vertex r ① do
3      A := ∅ and B = {r} # Build an M-alternating tree from r.
4      while there exists uv ∈ E with u ∈ B and v ∉ A ∪ B do
5          if v is M-covered then
6              Use uv to extend T
7          else
8              Use uv to augment M
9              break # Terminate the inner loop.
10     if r is still M-exposed ② then
11         return There is no perfect matching # T is a frustrated tree.
12 return Perfect matching M
```

## Theorem

The algorithm decides whether a given bipartite graph $G$ has a perfect matching and find one if exists. The algorithm calls $O(n)$ augmenting operations and $O(n^2)$ extending operations.

1. Actually, it suffices to once iterate over all vertices.
2. That is, the augmentation was no applied.

# Shrinking odd circuits

## Definition

Let $C$ be an odd circuit in $G$. The graph $G \times C$ has vertices $(V(G) \setminus V(C)) \cup \{c'\}$ where $c'$ is a new vertex and edges ①

- $E(G)$ with both end-vertices in $V(G) \setminus V(C)$ and
- and $uc'$ for every edge $uv$ with $u \notin V(C)$ and $v \in V(C)$.

Edges $E(C)$ are removed.

## Proposition

- $\mathrm{defic}(G) \leq \mathrm{defic}(G \times C)$
- There exists a graph $G$ and an odd cycle $C$ such that $\mathrm{defic}(G) < \mathrm{defic}(G \times C)$

## Remarks

- To find a maximum matching in $G$, it is not sufficient to find a maximum matching in $G \times C$ and extended by edges of $C$.
- We will contract only odd cycles on our alternating tree
- $G'$, $M'$ a $T'$ denotes graph, matching, and alternating tree obtained by a sequence of contractions

1. Formally, $E(G \times C) = \{uv;\ uv \in E(G),\ u, v \in V(G) \setminus V(C)\} \cup \{uc';\ \exists v \in V(C) :\ uv \in E(G), u \in V(G) \setminus V(C)\}$.

## Use $uv$ to shrink and update $M'$ and $T'$

Input: A matching $M'$ of a graph $G'$, an $M'$-alternating tree $T'$, edge $uv \in E'$ such that $u, v \in B'$

Action: Let $C$ be the circuit formed by $uv$ together with the path in $T'$ from $u$ to $v$.
Replace

- $G'$ by $G' \times C$
- $M'$ by $M' \setminus E(C)$
- $T$ by the tree having edge-set $E(T) \setminus E(C)$
- $A' := A' \setminus V(C)$
- $B' := B' \setminus V(C) \cup \{c'\}$ where $c'$ is a new pseudo-vertex

## Pozorování

- $T'$ je $M'$-alternating tree on vertices $A'$ a $B'$
- $A'$ contains only original vertices of $G$ (no pseudo-vertex)
- Odd components in $G' \setminus A'$ corresponds to odd vertices of $G \setminus A'$
- If $T'$ is a $M'$-frustrated tree in $G'$, then $G$ has no perfect matching

## Algorithm

```
1  Init: M := ∅
2  while G contains an M-exposed vertex r do
3      M′ = M, G′ = G and T = ({r}, ∅)
4      while there exists uv ∈ E(G′) with u ∈ B and v ∉ A do
5          if v ∈ B then
6              Use uv to shrink and update M′ and T
7          else if v is M′-covered then
8              Use uv to extend T
9          else
10             Use uv to augment M′
11             Extend M′ to a matching M of G
12             break # Terminate the inner loop.
13     if r is still M-exposed then
14         return There is no perfect matching
15 return Perfect matching M
```

# Maximum matching in general graphs

## Simple algorithm for implementation

Build an alternating tree from every uncovered vertex one by one.

- If an augmenting path is found, augment matching.
- If a frustrated tree is found, do nothing.

## Algorithm which calculates $A$ for Tutte-Berge formula

```
1  Init: M, Â, B̂ := ∅
2  for u ∈ V do
3      if u is not M-covered then
4          Build an M-altenating tree rooted in u on vertices A and B
5          if An augmenting path is found then
6              Augment matching M
7          else
8              Â := Â ∪ A
9              B̂ := B̂ ∪ B
10             G := G \ (A ∪ B)
11 return Maximal matching M
```

Observe that $V(G) - 2|M| = \mathrm{oc}(G \setminus \hat{A}) - |\hat{A}|$.

## Complexity

$\mathcal{O}(n)$ alternating trees are build and building one tree requires

- $\mathcal{O}(m)$ edges $uv$ are tested whether $u \in B$ and $v \notin A$
- $\mathcal{O}(n)$ edges extends the tree
- $\mathcal{O}(n)$ cycles are contracted
- The sum of length of contracted cycles is $\mathcal{O}(n)$
- The complexity of our algorithm will be $\mathcal{O}(nm\alpha(n))$ but we need to handle shrinking efficiently

## Speed up (Micali, Vazirani)

Basic idea:

- In every iteration, build alternating trees from all uncovered trees simultaneously
- Ihe number of iterations is $\mathcal{O}\left(\sqrt{n}\right)$ instead of $\mathcal{O}(n)$
- Total complexity is $\mathcal{O}\left(m\sqrt{n}\right)$

# Minimum spanning tree in a graph

## Problem

For a graph $G$ given by a list of edges sorted by their weights find a minimum spanning tree, i.e. tree covering all vertices.

## Algorithm

Start with the empty set of edges $T$ and process all edges $uv$ in increasing weight. If $u$ and $v$ belong to a different components of $(V, T)$, then add $uv$ into $T$.

## Question

How to determine whether $u$ and $v$ belong to different components?

## Union-Find problem

- Init: Every element (vertex) $u$ belong to the set $\{u\}$
- Union(u,v): Merge sets containing $u$ a $v$
- Find(u): Find a set containing $u$

# Disjoint-set data structure

## Forest

- A forest has one vertex for every element
- One of the forest corresponds to one set
- Every vertex $u$ stores its parent $p[u]$ in the forest
- The parent of a root of a tree is NULL $\Rightarrow$ initialize $p[u] = NULL$ for every vertices
- Every root stores the size of its tree

## Union(u,v)

- Find roots $u'$, $v'$ of trees containing $u$, $v$
- If $u'$ contains more element than $v'$, then $u'$ become the parent of $v'$

## Find(u): Find the root of $u$

- Find the root $u'$ of $u$
- For all vertices of the path from $u$ do $u'$ change the parent to be $u'$ (except $u'$)
- The amortized complexity is $O(\alpha(n))$

# Tasks for next week

## The second assignemnt

Finish Goldberg algorithm

## The third assignment

- Understand algorithm and its correctness
- Design data representation
- Find invariants and tests
- Try to implement finding maximum matching in bipartite graphs

## Literature


- Cunningham, Cook, Pulleyblank, Schrijver: Combinatorial optimization, John Wiley & Sons, 1997 (book chapter)
- Jan Vondrák: Polyhedral techniques in combinatorial optimization, 2010 (lecture notes) `https://theory.stanford.edu/~jvondrak/CS369P/lec4.pdf`
- Michel X. Goemans: Combinatorial Optimization (lecture notese) `http://math.mit.edu/~goemans/18433S15/matching-notes.pdf` `http://math.mit.edu/~goemans/18433S15/matching-nonbip-notes.pdf`
- Uri Zwick: Lecture notes on: Maximum matching in bipartite and non-bipartite graphs `https://www.cs.tau.ac.il/~zwick/grad-algo-0910/match.pdf`
- Visualization: `https://algorithms.discrete.ma.tum.de/graph-algorithms/matchings-blossom-algorithm/index_en.html`