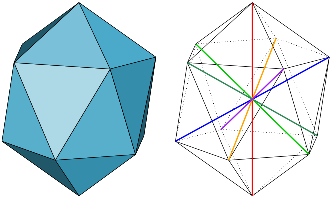
The problem of lines spanning the same angle

Problem: What is the maximum number of lines in the d-dimensional Euclidean space $\mathbb{R}^d$ so that between every pair of these lines is the same angle φ?

Examples:

In $\mathbb{R}^2$ there are 3 lines with $\varphi = 60^\circ$.

In $\mathbb{R}^3$ there are 6 lines — connecting the opposite vertices of the icosahedron.



Theorem: In $\mathbb{R}^d$ at most $\binom{d+1}{2}$ lines may span the same angle.

Proof: Assume that n such lines exist. Choose vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ by one each from each line s.t. these vectors are of unit length.

We get:

$$\langle \boldsymbol{v}_i | \boldsymbol{v}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ \cos\varphi & \text{otherwise} \end{cases}$$

We show that the matrices $\boldsymbol{v}_1\boldsymbol{v}_1^T, \boldsymbol{v}_2\boldsymbol{v}_2^T, \ldots, \boldsymbol{v}_n\boldsymbol{v}_n^T \in \mathbb{R}^{d\times d}$ are linearly independent. Then $n \le \binom{d+1}{2}$ as the dimension of the space of symmetric matrices from $\mathbb{R}^{d\times d}$ is at most $\binom{d+1}{2}$.

Linear independence of matrices $\boldsymbol{v}_1\boldsymbol{v}_1^T, \boldsymbol{v}_2\boldsymbol{v}_2^T, \ldots, \boldsymbol{v}_n\boldsymbol{v}_n^T$

Assume that $\sum_{i=1}^{n} a_i \boldsymbol{v}_i \boldsymbol{v}_i^T = \mathbf{0}$ (the $d \times d$ matrix full of zeroes).

Then for any $j \in \{1, \ldots, n\}$: $0 = \boldsymbol{v}_j^T \mathbf{0} \boldsymbol{v}_j = \boldsymbol{v}_j^T \left(\sum_{i=1}^{n} a_i \boldsymbol{v}_i \boldsymbol{v}_i^T \right) \boldsymbol{v}_j =$

$$= \sum_{i=1}^{n} a_i \boldsymbol{v}_j^T \boldsymbol{v}_i \boldsymbol{v}_i^T \boldsymbol{v}_j = \sum_{i=1}^{n} a_i \langle \boldsymbol{v}_i | \boldsymbol{v}_j \rangle^2 = a_j + \cos^2\varphi \sum_{i \neq j} a_i$$

These conditions on $a_1, \ldots, a_n$ written as a system of equations:

$$\begin{pmatrix} 1 & \cos^2\varphi & \ldots & \cos^2\varphi \\ \cos^2\varphi & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \cos^2\varphi \\ \cos^2\varphi & \ldots & \cos^2\varphi & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The matrix of this system is regular, hence $a_1 = \ldots = a_n = 0$.

Therefore $\boldsymbol{v}_1\boldsymbol{v}_1^T, \boldsymbol{v}_2\boldsymbol{v}_2^T, \ldots, \boldsymbol{v}_n\boldsymbol{v}_n^T$ are linearly independent.