Conics and quadrics

[Wiki:] „*Conic is a curve obtained as the intersection of the surface of a cone with a plane*“

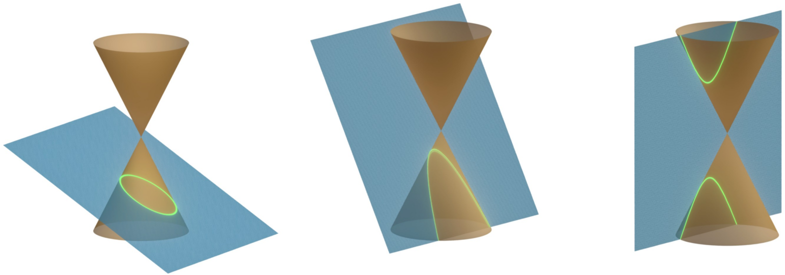
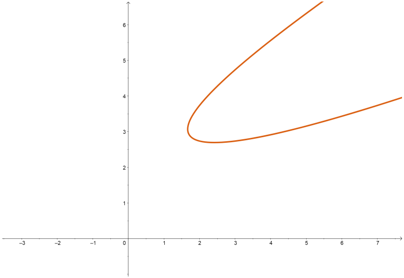


Fig.: en.wikipedia.org/wiki/Conic_section

Conics and quadrics

Definition: *Conic* is the set of solutions of a homogeneous equation with real polynomial od degree two in two unknowns, i.e. :
$\{\boldsymbol{x} \in \mathbb{R}^2 : a_{1,1}x_1^2 + a_{1,2}x_1x_2 + a_{2,2}x_2^2 + b_1x_1 + b_2x_2 + c = 0\}$



Left: $(x_1 - 2)^2 + 4(x_2 - 2)^2 = 16$ ... ellipse

Right: $2x_1^2 - 8x_1x_2 + 8x_2^2 + (8\sqrt{5} - 4)x_1 - (16\sqrt{5} + 2)x_2 + 50 = 0$
... ellipse ? parabola ? hyperbola

Fig.: `www.geogebra.org/graphing`

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The same written with a matrix $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ and a vector $\boldsymbol{b} \in \mathbb{R}^2$:

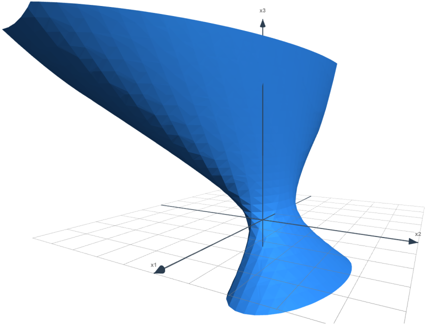
$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x} + c = 0$$

(Either choose $a_{2,1} = 0$ or split the coefficient by x_1x_2 symetrically.)
Definition: For a matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, vector $\boldsymbol{b} \in \mathbb{R}^d$ and a scalar $c \in \mathbb{R}$ the *quadrics* is the set $\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x} + c = 0\}$.

In the nondegenerate case we get a $(d-1)$-dimensional surface in d-dimensional space.

$$\begin{gathered} x_1^2 + 3x_2^2 - x_3^2 \\ + x_1x_2 + x_1x_3 + 4x_2x_3 \\ + 4x_1 + 5x_2 + 3x_3 + 3 = 0 \end{gathered}$$

Fig.: `www.math3d.org`



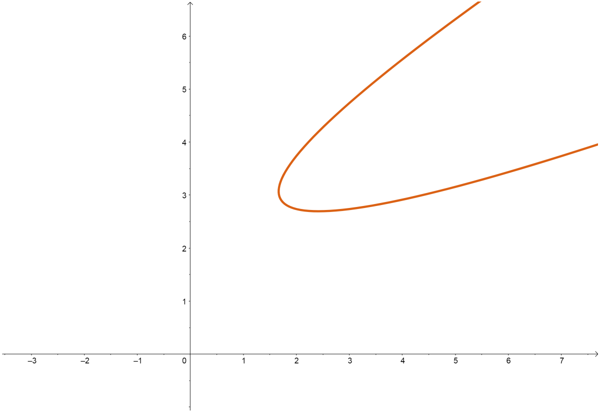
Applications

- planetary motion in astronomy — ellipses
- construction of optical mirrors and microphones — parabolic surfaces
- linear programming — ellipsoid method
- physics — calculation of stress inside a body or description of a rotational motion of rigid bodies (e.g. gyroscopes)
- statistics — principal components analysis e.g. to reduce the size of large data files without significant data loss
- informatics — pattern recognition, neural networks
- electronics — design and analysis of circuit behavior
- arithmetic, number theory, . . .

Quadric transformation

- Given $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x} + c = 0$ with a symmetric $\boldsymbol{A}$ w.r.t. X.
- We find basis Y so that $[id]_{Y,X}$ is orthogonal and $\boldsymbol{A}' = [id]_{Y,X}^T\boldsymbol{A}[id]_{Y,X}$ is diagonal.
- Substitute (first) $\boldsymbol{x} = [id]_{Y,X}\boldsymbol{y}$, thus get $\boldsymbol{y}^T\boldsymbol{A}'\boldsymbol{y} + \boldsymbol{b}'^T\boldsymbol{y} + c' = 0$ for $\boldsymbol{b}' = [id]_{Y,X}^T\boldsymbol{b}$ and $c = c'$.
  Isometry geometrically means a rotation of the coordinates.
- For each $a'_{i,i} \neq 0$ subtitute (second) $y_i = z_i - \frac{b'_i}{2a'_{i,i}}$.
  This is a shift in the origin of the coordinate system so that
  $$a'_{i,i}y_i^2 + b'_i y_i = a'_{i,i}\left(z_i - \frac{b'_i}{2a'_{i,i}}\right)^2 + b'_i\left(z_i - \frac{b'_i}{2a'_{i,i}}\right) = a'_{i,i}z_i^2 - \frac{b_i'^2}{4a'_{i,i}}$$
  that is, nonzero quadratic terms absorb their linear terms.
  We get $\boldsymbol{z}^T\boldsymbol{A}''\boldsymbol{z} + \boldsymbol{b}''^T\boldsymbol{z} + c'' = 0$ where $\boldsymbol{A}'' = \boldsymbol{A}'$,
  $$b''_i = \begin{cases} 0 & \text{for } a''_{i,i} \neq 0 \\ b'_i & \text{for } a''_{i,i} = 0 \end{cases} \quad \text{and} \quad c'' = c' - \sum_{a''_{i,i}\neq 0} \frac{b_i^2}{4a_{i,i}}.$$
- Now easily derive shape, axes, center and other parameters.

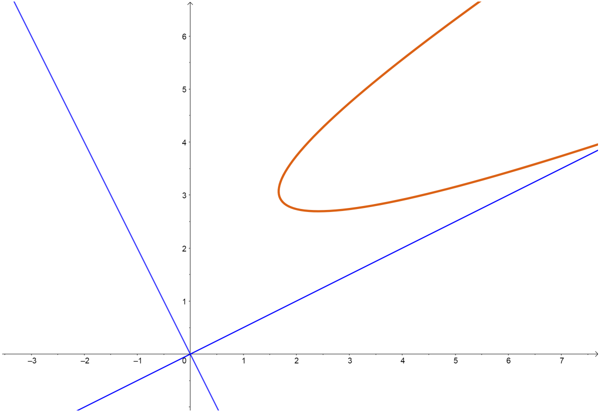
Example of a transformation



$$2x_1^2 - 8x_1x_2 + 8x_2^2 + (8\sqrt{5} - 4)x_1 - (16\sqrt{5} + 2)x_2 + 50 = 0$$

Diagonalize matrix $\begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$ to find the new basis.

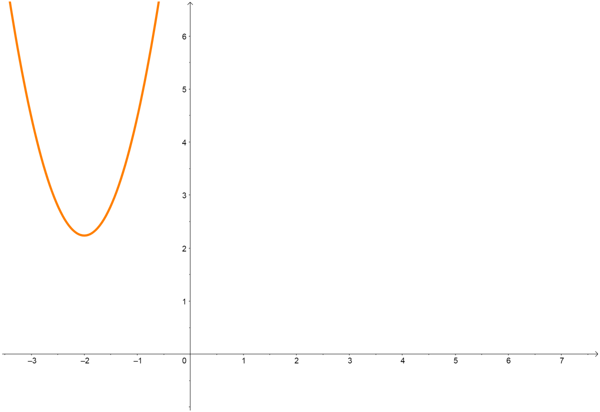
Example of a transformation



Use orthogonal $[id]_{Y,X} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$,

to perform only an isometry (axes rotation)

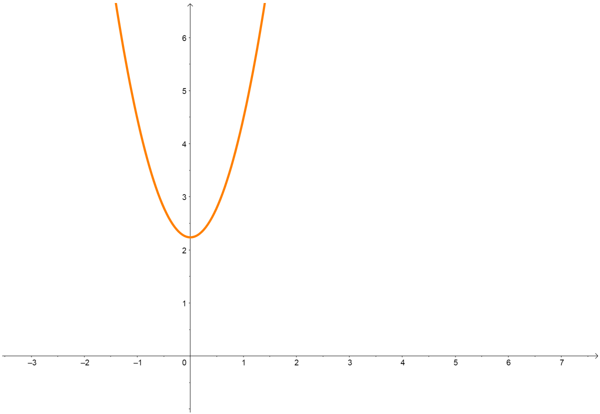
Example of a transformation



$$10y_1^2 - 2\sqrt{5}y_2 + 40y_1 + 50 = 0$$

Substitute $y_1 = z_1 - 2$, $y_2 = z_2$ (horizontal shift)

Example of a transformation



The resulting parabola: $10z_1^2 - 2\sqrt{5}z_2 + 10 = 0$

(It is also possible to make a vertical shift to the origin.)