Questions to understand the topic of the lecture

- Is it true that no quadratic form over a vector space of characteristic two can be diagonalized?
- If there exists a symmetric bilinear form f that corresponds to a given quadratic form g, then is f unique?
- How the coefficients of an analytic expression change if we change the basis?
- Is it true that if a symmetric matrix $\boldsymbol{A}$ can be diagonalized by $\boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R}$, then $\boldsymbol{R}$ can always be chosen upper triangular?
- Is it true that when a quadratic form g on V over $\mathbb{R}$ has diagonal matrix with some 1 and some -1, then there exist vectors $\boldsymbol{u}, \boldsymbol{w} \in V$ such that $g(\boldsymbol{u}) < 0 < g(\boldsymbol{w})$?

Bilinear and quadratic forms

Definition: Let V be a vector space over a field $\mathbb{K}$ and let a mapping $f : V \times V \to \mathbb{K}$ satisfies:

- $\forall \boldsymbol{u}, \boldsymbol{v} \in V, \forall a \in \mathbb{K} : f(a\boldsymbol{u}, \boldsymbol{v}) = f(\boldsymbol{u}, a\boldsymbol{v}) = af(\boldsymbol{u}, \boldsymbol{v})$
- $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V : f(\boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w}) = f(\boldsymbol{u}, \boldsymbol{w}) + f(\boldsymbol{v}, \boldsymbol{w})$
- $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V : f(\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}) = f(\boldsymbol{u}, \boldsymbol{v}) + f(\boldsymbol{u}, \boldsymbol{w})$

Then f is called a *bilinear form* on V.

A bilinear form is *symmetric* if $\forall \boldsymbol{u}, \boldsymbol{v} \in V : f(\boldsymbol{u}, \boldsymbol{v}) = f(\boldsymbol{v}, \boldsymbol{u})$.

A mapping $g : V \to \mathbb{K}$ is called a *quadratic form*, if there exists a bilinear form f such that $g(\boldsymbol{u}) = f(\boldsymbol{u}, \boldsymbol{u})$ for all $\boldsymbol{u} \in V$.

Examples: Any inner product on a space over $\mathbb{R}$, *but not over* $\mathbb{C}$!

For $V = \mathbb{Z}_5^2$, a bilinear form:
$f(\boldsymbol{u}, \boldsymbol{v}) = u_1 v_1 + 2u_1 v_2 + 4u_2 v_1 + 3u_2 v_2$

The corresponding quadratic form:
$g(\boldsymbol{u}) = f(\boldsymbol{u}, \boldsymbol{u}) = u_1 u_1 + 2u_1 u_2 + 4u_2 u_1 + 3u_2 u_2 = u_1^2 + u_1 u_2 + 3u_2^2$

Matrices of forms

Definition: Let V be a vector space over a field $\mathbb{K}$ and let $X = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ be its basis. The *matrix of a bilinear form f w.r.t. the basis X* is the matrix $\boldsymbol{B}$ defined as $b_{i,j} = f(\boldsymbol{v}_i, \boldsymbol{v}_j)$.
The matrix of a quadratic form g is the matrix of a symmetric bilinear form f corresponding to g, if such symmetric f exists.
Example: For $V = \mathbb{Z}_5^2$, and the canonical basis K, the bilinear form $f(\boldsymbol{u}, \boldsymbol{v}) = u_1v_1 + 2u_1v_2 + 4u_2v_1 + 3u_2v_2$ has matrix $\boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$
and $g(\boldsymbol{u}) = u_1^2 + u_1u_2 + 3u_2^2$ has matrix $\boldsymbol{B} = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}$

On $V = \mathbb{Z}_2^2$ the quadratic form $g(\boldsymbol{u}) = u_1u_2$ corresponds e.g. to the bilinear form with matrix $\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ but to no symmetric.

Matrices of forms

Definition: Let V be a vector space over a field $\mathbb{K}$ and let $X = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ be its basis. The *matrix of a bilinear form f w.r.t. the basis X* is the matrix $\boldsymbol{B}$ defined as $b_{i,j} = f(\boldsymbol{v}_i, \boldsymbol{v}_j)$.
The matrix of a quadratic form g is the matrix of a symmetric bilinear form f corresponding to g, if such symmetric f exists.

Observation: $b_{i,j} = f(\boldsymbol{v}_i, \boldsymbol{v}_j) = \frac{1}{2}(g(\boldsymbol{v}_i + \boldsymbol{v}_j) - g(\boldsymbol{v}_i) - g(\boldsymbol{v}_j))$

Proof: $g(\boldsymbol{v}_i + \boldsymbol{v}_j) = f(\boldsymbol{v}_i + \boldsymbol{v}_j, \boldsymbol{v}_i + \boldsymbol{v}_j)$

$$= f(\boldsymbol{v}_i, \boldsymbol{v}_i) + f(\boldsymbol{v}_i, \boldsymbol{v}_j) + f(\boldsymbol{v}_j, \boldsymbol{v}_i) + f(\boldsymbol{v}_j, \boldsymbol{v}_j)$$

$$g(\boldsymbol{v}_i + \boldsymbol{v}_j) - g(\boldsymbol{v}_i) - g(\boldsymbol{v}_j) = f(\boldsymbol{v}_i, \boldsymbol{v}_j) + f(\boldsymbol{v}_j, \boldsymbol{v}_i)$$

Observation: The use of matrices of forms:

$$f(\boldsymbol{u}, \boldsymbol{v}) = [\boldsymbol{u}]_X^T \boldsymbol{B} [\boldsymbol{v}]_X, \quad g(\boldsymbol{u}) = [\boldsymbol{u}]_X^T \boldsymbol{B} [\boldsymbol{u}]_X.$$

Proof: When $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{v}_i$ and $\boldsymbol{w} = \sum_{j=1}^{n} b_j \boldsymbol{v}_j$, then

$$f(\boldsymbol{u}, \boldsymbol{w}) = f\left(\sum_{i=1}^{n} a_i \boldsymbol{v}_i, \sum_{j=1}^{n} b_j \boldsymbol{v}_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i f(\boldsymbol{v}_i, \boldsymbol{v}_j) b_j = [\boldsymbol{u}]_X^T \boldsymbol{B} [\boldsymbol{w}]_X$$

Definition: The *analytic expression* of a bilinear form f over $\mathbb{K}^n$ with matrix $\boldsymbol{B}$ is the homogeneous polynomial

$$f((x_1,\dots,x_n)^T,(y_1,\dots,y_n)^T) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{i,j}x_iy_j$$

...analogously for quadratic forms and/or relative to a basis X.

Observation: Let $\boldsymbol{B}$ be a matrix of a b/q form w.r.t. a basis X. Then $[id]_{YX}^T\boldsymbol{B}[id]_{YX}$ is the matrix of the same form w.r.t. Y.

Proof: $[\boldsymbol{u}]_X = [id]_{YX}[\boldsymbol{u}]_Y$, $[\boldsymbol{v}]_X = [id]_{YX}[\boldsymbol{v}]_Y$,

$$f(\boldsymbol{u},\boldsymbol{v}) = [\boldsymbol{u}]_X^T\boldsymbol{B}[\boldsymbol{v}]_X = ([id]_{YX}[\boldsymbol{u}]_Y)^T\boldsymbol{B}[id]_{YX}[\boldsymbol{v}]_Y$$

$$= [\boldsymbol{u}]_Y^T[id]_{YX}^T\boldsymbol{B}[id]_{YX}[\boldsymbol{v}]_Y.$$

Diagonalization of forms

Theorem: If g is a quadratic form on a vector space V of finite dimension n over a field $\mathbb{K}$ other characteristics than 2, then the form g allows a diagonal matrix $\boldsymbol{B}$ w.r.t. a suitable basis X.

(holds also for symmetric bilinear forms)

Rephrased in terms of matrices:

Theorem: For any symmetric matrix $\boldsymbol{A} \in \mathbb{K}^{n\times n}$ with $\mathrm{char}(\mathbb{K}) \neq 2$ there is a regular matrix $\boldsymbol{R}$ such that $\boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R}$ is diagonal.

Compare with the diagonalization of *real* symmetric matrices of linear maps — $\boldsymbol{R}$ could indeed be *orthogonal*: $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$, hence $\boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R} = \boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}$. Columns of $\boldsymbol{R}$ (ON basis) are *principal axes*.

Example: No way to diagonalize $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ over $\mathbb{Z}_2$,

but over $\mathbb{Z}_3$ it is possible: $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

Theorem: For any symmetric matrix $\boldsymbol{A} \in \mathbb{K}^{n \times n}$ with $\mathrm{char}(\mathbb{K}) \neq 2$ there is a regular matrix $\boldsymbol{R}$ such that $\boldsymbol{R}^T \boldsymbol{A} \boldsymbol{R}$ is diagonal.

Proof: By induction on n.

Denote $\boldsymbol{A} = \boldsymbol{A}_n = \left[\begin{array}{c|c} \alpha & \boldsymbol{a}^T \\ \hline \boldsymbol{a} & \tilde{\boldsymbol{A}} \end{array}\right]$.

If $\alpha \neq 0$, let $\boldsymbol{P}_n = \left[\begin{array}{c|c} 1 & -\frac{1}{\alpha}\boldsymbol{a}^T \\ \hline \mathbf{0} & \boldsymbol{I}_{n-1} \end{array}\right]$.

$$\text{Then } \boldsymbol{P}_n^T \boldsymbol{A}_n \boldsymbol{P}_n = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline -\frac{1}{\alpha}\boldsymbol{a} & \boldsymbol{I}_{n-1} \end{array}\right] \cdot \left[\begin{array}{c|c} \alpha & \boldsymbol{a}^T \\ \hline \boldsymbol{a} & \tilde{\boldsymbol{A}} \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & -\frac{1}{\alpha}\boldsymbol{a}^T \\ \hline \mathbf{0} & \boldsymbol{I}_{n-1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \alpha & \boldsymbol{a}^T \\ \hline \mathbf{0} & -\frac{1}{\alpha}\boldsymbol{a}\boldsymbol{a}^T + \tilde{\boldsymbol{A}} \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & -\frac{1}{\alpha}\boldsymbol{a}^T \\ \hline \mathbf{0} & \boldsymbol{I}_{n-1} \end{array}\right] = \left[\begin{array}{c|c} \alpha & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{A}_{n-1} \end{array}\right]$$

where $\boldsymbol{A}_{n-1} = \tilde{\boldsymbol{A}} - \frac{1}{\alpha}\boldsymbol{a}\boldsymbol{a}^T$ is symmetric.

Theorem: For any symmetric matrix $\boldsymbol{A} \in \mathbb{K}^{n\times n}$ with $\mathrm{char}(\mathbb{K}) \neq 2$ there is a regular matrix $\boldsymbol{R}$ such that $\boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R}$ is diagonal.

Proof: By induction on n.

Denote $\boldsymbol{A} = \boldsymbol{A}_n = \left[\begin{array}{c|c} \alpha & \boldsymbol{a}^T \\ \hline \boldsymbol{a} & \tilde{\boldsymbol{A}} \end{array}\right]$.

If $\alpha \neq 0$, let $\boldsymbol{P}_n = \left[\begin{array}{c|c} 1 & -\frac{1}{\alpha}\boldsymbol{a}^T \\ \hline \mathbf{0} & \boldsymbol{I}_{n-1} \end{array}\right]$. Then $\boldsymbol{P}_n^T\boldsymbol{A}_n\boldsymbol{P}_n = \left[\begin{array}{c|c} \alpha & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{A}_{n-1} \end{array}\right]$,

with $\boldsymbol{A}_{n-1}$ symmetric. By induction hypothesis there exists $\boldsymbol{R}_{n-1}$ for $\boldsymbol{A}_{n-1}$. We choose $\boldsymbol{R}_n = \boldsymbol{P}_n \cdot \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1} \end{array}\right]$

Then $\boldsymbol{R}_n^T\boldsymbol{A}_n\boldsymbol{R}_n = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1}^T \end{array}\right] \cdot \boldsymbol{P}_n^T\boldsymbol{A}_n\boldsymbol{P}_n \cdot \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1} \end{array}\right]$

$= \left[\begin{array}{c|c} \alpha & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1}^T\boldsymbol{A}_{n-1}\boldsymbol{R}_{n-1} \end{array}\right]$ is diagonal.

Theorem: For any symmetric matrix $\boldsymbol{A} \in \mathbb{K}^{n\times n}$ with $\mathrm{char}(\mathbb{K}) \neq 2$ there is a regular matrix $\boldsymbol{R}$ such that $\boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R}$ is diagonal.

Proof: By induction on n.

Denote $\boldsymbol{A} = \boldsymbol{A}_n = \left(\begin{array}{c|c} \alpha & \boldsymbol{a}^T \\ \hline \boldsymbol{a} & \tilde{\boldsymbol{A}} \end{array}\right)$.

If $\alpha \neq 0$, let $\boldsymbol{P}_n = \left(\begin{array}{c|c} 1 & -\frac{1}{\alpha}\boldsymbol{a}^T \\ \hline \boldsymbol{0} & \boldsymbol{I}_{n-1} \end{array}\right)$. Then $\boldsymbol{P}_n^T\boldsymbol{A}_n\boldsymbol{P}_n = \left(\begin{array}{c|c} \alpha & \boldsymbol{0}^T \\ \hline \boldsymbol{0} & \boldsymbol{A}_{n-1} \end{array}\right)$,

with $\boldsymbol{A}_{n-1}$ symmetric. By induction hypothesis there exists $\boldsymbol{R}_{n-1}$ for $\boldsymbol{A}_{n-1}$. We choose $\boldsymbol{R}_n = \boldsymbol{P}_n \cdot \left(\begin{array}{c|c} 1 & \boldsymbol{0}^T \\ \hline \boldsymbol{0} & \boldsymbol{R}_{n-1} \end{array}\right)$

Then $\boldsymbol{R}_n^T\boldsymbol{A}_n\boldsymbol{R}_n$ is diagonal.

Example: $\mathbb{K} = \mathbb{Z}_3$, $\boldsymbol{A}_3 = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, $\alpha = 2$, $\boldsymbol{P}_3 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$\boldsymbol{A}_2 = \tilde{\boldsymbol{A}} - \frac{1}{\alpha}\boldsymbol{a}\boldsymbol{a}^T = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}(2,1) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\boldsymbol{R}_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

$\boldsymbol{R}_3 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, $\boldsymbol{R}_3^T\boldsymbol{A}_3\boldsymbol{R}_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Theorem: For any symmetric matrix $\boldsymbol{A} \in \mathbb{K}^{n\times n}$ with $\mathrm{char}(\mathbb{K}) \neq 2$ there is a regular matrix $\boldsymbol{R}$ such that $\boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R}$ is diagonal.

Proof: By induction on n.

Denote $\boldsymbol{A} = \boldsymbol{A}_n = \left[\begin{array}{c|c} \alpha & \boldsymbol{a}^T \\ \hline \boldsymbol{a} & \tilde{\boldsymbol{A}} \end{array}\right]$.

If $\alpha \neq 0$, let $\boldsymbol{P}_n = \left[\begin{array}{c|c} 1 & -\frac{1}{\alpha}\boldsymbol{a}^T \\ \hline \mathbf{0} & \boldsymbol{I}_{n-1} \end{array}\right]$. Then $\boldsymbol{P}_n^T\boldsymbol{A}_n\boldsymbol{P}_n = \left[\begin{array}{c|c} \alpha & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{A}_{n-1} \end{array}\right]$,

with $\boldsymbol{A}_{n-1}$ symmetric. By induction hypothesis there exists $\boldsymbol{R}_{n-1}$ for $\boldsymbol{A}_{n-1}$. We choose $\boldsymbol{R}_n = \boldsymbol{P}_n \cdot \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1} \end{array}\right]$

Then $\boldsymbol{R}_n^T\boldsymbol{A}_n\boldsymbol{R}_n$ is diagonal.

If $\alpha = 0$ but $\boldsymbol{a} \neq \mathbf{0}$, then $a_{i,1} \neq 0$ for some i. Use the elementary matrix $\boldsymbol{E}$ for adding the i-th column to the first. Take $\boldsymbol{A}' = \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E}$ instead of $\boldsymbol{A}$. As $\alpha' = 2a_{i,1} \neq 0$, we may follow the previous case.

If $\alpha = 0$ and $\boldsymbol{a} = \mathbf{0}$, then let $\boldsymbol{A}_{n-1} = \tilde{\boldsymbol{A}}$ and get $\boldsymbol{R}_n = \left[\begin{array}{c|c} 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1} \end{array}\right]$.

Methods of diagonalization

- Real symmetric matrices can be diagonalized with orthonormal eigenvectors.
- By Gaussian elimination — we perform each operation *simultaneously* on both rows and columns.

Observation: If $\boldsymbol{A}$ is symmetric then $\boldsymbol{A}' = \boldsymbol{E}^T\boldsymbol{A}\boldsymbol{E}$ is symmetric too.

Corollary: Lower triangular $\boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R}$ is diagonal.

Example:

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{\text{row}}{\overset{\text{II-I}}{\sim}} \left(\begin{array}{ccc|ccc} 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \underset{\text{col.}}{\overset{\text{II-I}}{\sim}} \left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\overset{\text{III+I}}{\sim} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right) \overset{\text{III-II}}{\sim} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{array}\right)$$

The diagonal matrix $\boldsymbol{R}^T\boldsymbol{A}\boldsymbol{R}$ is on the left.
On the right is the matrix of *row* operations, i.e. $\boldsymbol{R}^T$.

Sylvester's law of inertia

Theorem: Every quadratic form on a finitely generated *real* vector space allows a diagonal matrix with only 1, -1 and 0.
Moreover, all such diagonal matrices corresponding to the same form have the same number of 1's and the same number of -1's.

Definition: Let a real quadratic form g is represented by a diagonal matrix $\boldsymbol{B}$ containing only 1, -1 and 0.
The *signature* of the form g is the triple $(\#1, \# - 1, \#0)$, counted along the diagonal of the matrix $\boldsymbol{B}$.

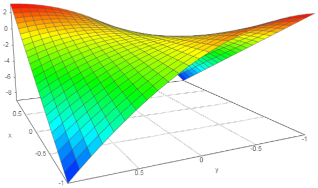
Sylvester's law of inertia

Theorem: Every quadratic form on a finitely generated *real* vector space allows a diagonal matrix with only 1, -1 and 0.
Moreover, all such diagonal matrices corresponding to the same form have the same number of 1's and the same number of -1's.

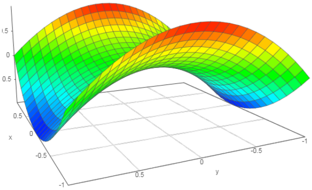
Example: $g : \mathbb{R}^2 \to \mathbb{R}$ given by $\boldsymbol{B} = \begin{pmatrix} 0 & 3 \\ 3 & -3 \end{pmatrix}$ w.r.t. K.

The matrix of g w.r.t. the basis: $X = \{(\frac{2}{3}, \frac{1}{3})^T, (-\frac{1}{3}, \frac{1}{3})^T\}$ is

$$\boldsymbol{B}' = [id]_{XK}^T \, \boldsymbol{B} \, [id]_{XK} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

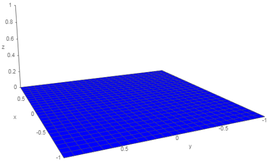


$6x_1x_2 - 3x_2^2$



$x_1^2 - x_2^2$

The six cases of diagonalized quadratic forms $\mathbb{R}^2 \to \mathbb{R}$



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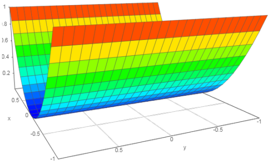
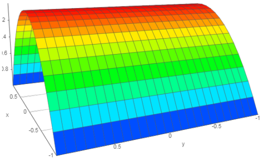
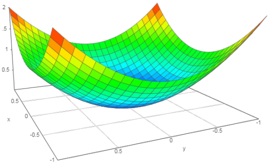


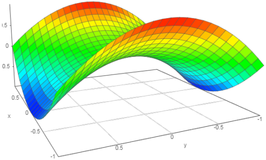
x_1^2



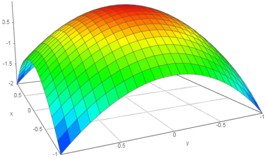
$-x_1^2$



$x_1^2 + x_2^2$



$x_1^2 - x_2^2$



$-x_1^2 - x_2^2$

(ordered by the rank and then 1 before -1)

Sylvester's law of inertia

Theorem: Every quadratic form on a finitely generated *real* vector space allows a diagonal matrix with only 1, -1 and 0.
Moreover, all such diagonal matrices corresponding to the same form have the same number of 1's and the same number of -1's.
Proof:
1. Existence: Let $\boldsymbol{B}$ be the matrix of the form. w.r.t. some basis Y.
Real symmetric matrices can be diagonalized, i.e. any $\boldsymbol{B} = \boldsymbol{R}^T \boldsymbol{D} \boldsymbol{R}$ for a regular $\boldsymbol{R}$.

Split $\boldsymbol{D}$ as $\boldsymbol{S}^T \boldsymbol{D}' \boldsymbol{S}$ where $d_{i,i} \begin{cases} = 0 & d'_{i,i} = 0, \quad s_{i,i} = 1 \\ > 0 & d'_{i,i} = 1, \quad s_{i,i} = \sqrt{d_{i,i}} \\ < 0 & d'_{i,i} = -1, s_{i,i} = \sqrt{-d_{i,i}} \end{cases}$

Now $\boldsymbol{SR}$ is regular and $\boldsymbol{B} = (\boldsymbol{SR})^T \boldsymbol{D}' \boldsymbol{SR}$.
Choose the basis X, the coordinates of vectors of X w.r.t. Y are the columns of $\boldsymbol{SR}$, i.e. $[id]_{X,Y} = \boldsymbol{SR}$ and also $[id]_{Y,X} = (\boldsymbol{SR})^{-1}$.
Now $[id]^T_{Y,X} \boldsymbol{B} [id]_{Y,X} = ((\boldsymbol{SR})^{-1})^T (\boldsymbol{SR})^T \boldsymbol{D}' \boldsymbol{SR} (\boldsymbol{SR})^{-1} = \boldsymbol{D}'$ is the desired diagonal matrix of the form.

Sylvester's law of inertia

Theorem: Every quadratic form on a finitely generated *real* vector space allows a diagonal matrix with only 1, -1 and 0. Moreover, all such diagonal matrices corresponding to the same form have the same number of 1's and the same number of -1's.

Example:

$$\boldsymbol{B} = \begin{pmatrix} 7 & -10 & -2 \\ -10 & 4 & 8 \\ -2 & 8 & -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 18 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \boldsymbol{R}^T\boldsymbol{D}\boldsymbol{R}$$

$$\boldsymbol{D} = \begin{pmatrix} 18 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{S}^T\boldsymbol{D}'\boldsymbol{S}$$

$$[id]_{X,Y} = \boldsymbol{S}\boldsymbol{R} = \begin{pmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} & -2\sqrt{2} & -\sqrt{2} \\ 1 & 2 & -2 \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

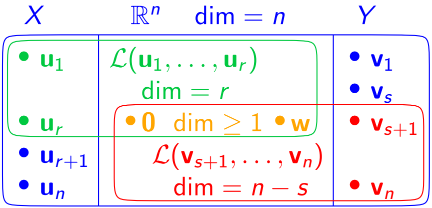
$$\boldsymbol{B} = \boldsymbol{R}^T\boldsymbol{D}\boldsymbol{R} = \boldsymbol{R}^T\boldsymbol{S}^T\boldsymbol{D}'\boldsymbol{S}\boldsymbol{R} = (\boldsymbol{S}\boldsymbol{R})^T\boldsymbol{D}'\boldsymbol{S}\boldsymbol{R} = [id]_{X,Y}^T\boldsymbol{D}'[id]_{X,Y}$$

$$\iff \qquad [id]_{Y,X}^T\boldsymbol{B}[id]_{Y,X} = \boldsymbol{D}'$$

2. Uniqueness of the numbers of 1's, -1's (and hence also 0's):
Let $X = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$, $Y = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ be two bases s.t. the corresponding matrices $\boldsymbol{B}$ and $\boldsymbol{B}'$ of the form g are diagonal with 1's, -1's and 0's ordered. s.t. 1's are first, then -1's and 0's are last.

As products with regular matrices $[id]_{XY}$ do not change the rank:
$\#0$'s in $\boldsymbol{B} = n - \mathrm{rank}(\boldsymbol{B}) = n - \mathrm{rank}(\boldsymbol{B}') = \#0$'s in $\boldsymbol{B}'$.

Let $r = \#1$'s in $\boldsymbol{B}$, $s = \#1$'s in $\boldsymbol{B}'$. If $r > s$, then consider the subspaces $\mathcal{L}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$ and $\mathcal{L}(\boldsymbol{v}_{s+1}, \ldots, \boldsymbol{v}_n)$. The sum of their dimensions $r + n - s$ exceeds n, hence they intersect nontrivially.



We use a fact from WT:
$\dim(U) + \dim(V) = \dim(U \cap V) + \dim(\mathcal{L}(U \cup V))$

LHS is strictly bigger than n,
$\dim(\mathcal{L}(U \cup V)) \leq \dim(\mathbb{R}^n) = n$
$\Longrightarrow \dim(U \cap V) \geq 1$

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Choose $\boldsymbol{w} \in (\mathcal{L}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r) \cap \mathcal{L}(\boldsymbol{v}_{s+1}, \ldots, \boldsymbol{v}_n)) \setminus \mathbf{0}$, thus
$[\boldsymbol{w}]_X = (x_1, \ldots, x_r, 0, \ldots, 0)^T$, $[\boldsymbol{w}]_Y = (0, \ldots, 0, y_{s+1}, \ldots, y_n)^T$.
Now $g(\boldsymbol{w}) = [\boldsymbol{w}]_X^T \boldsymbol{B} [\boldsymbol{w}]_X = x_1^2 + \cdots + x_r^2 > 0$ ($>$ as $\boldsymbol{w} \neq \mathbf{0}$), but
$g(\boldsymbol{w}) = [\boldsymbol{w}]_Y^T \boldsymbol{B}' [\boldsymbol{w}]_Y = -y_{s+1}^2 - \ldots - y_{\text{rank}(\boldsymbol{B}')}^2 \leq 0$, contradiction.
Therefore $r \not> s$, by symmetry also $s \not> r$, hence $r = s$.

Comments

Observation: Forms with *real* positive definite matrices are those that could be diagonalized into I_n

— compare Cholesky factorization $\boldsymbol{A} = \boldsymbol{U}^H \boldsymbol{U} = \boldsymbol{U}^T \boldsymbol{I}_n \boldsymbol{U}$.

Observation: An analogous statement for *complex symmetric* forms (other property than Hermitian!) yields diagonal matrices with 1's and 0's on the diagonal; including the inertia.