Special complex matrices

Definition: The *Hermitian transpose* of a complex matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ is the matrix $\boldsymbol{A}^H \in \mathbb{C}^{n\times m}$ where $(\boldsymbol{A}^H)_{i,j} = \overline{a_{j,i}}$.

Definition: A matrix $\boldsymbol{A}$ is *Hermitian* if $\boldsymbol{A} = \boldsymbol{A}^H$.

Definition: A matrix $\boldsymbol{A}$ is *unitary* if $\boldsymbol{A}^{-1} = \boldsymbol{A}^H$.

real	complex
transpose $\boldsymbol{A} \to \boldsymbol{A}^T$ $\begin{pmatrix} 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 \\ 3 \end{pmatrix}$	Hermitian transpose $\boldsymbol{A} \to \boldsymbol{A}^H$ $\begin{pmatrix} 1+i & -2i \end{pmatrix} \to \begin{pmatrix} 1-i \\ 2i \end{pmatrix}$
symmetric $\boldsymbol{A} = \boldsymbol{A}^T$ $\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$	Hermitian $\boldsymbol{A} = \boldsymbol{A}^H$ $\begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$
orthogonal $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$ $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$	unitary $\boldsymbol{A}^{-1} = \boldsymbol{A}^H$ $\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$

Properties

Observation: Hermitian matrices have real diagonal:
If $a_{i,i} = \overline{a_{i,i}}$, then $a_{i,i} \in \mathbb{R}$.

Observation: $(\boldsymbol{A}^H)^H = \boldsymbol{A}, \quad (\boldsymbol{AB})^H = \boldsymbol{B}^H \boldsymbol{A}^H$

Observation: If $\boldsymbol{A}$ is unitary then $\boldsymbol{A}^H$ is unitary.
$(\boldsymbol{A}^H)^H = \boldsymbol{A} = (\boldsymbol{A}^{-1})^{-1} = (\boldsymbol{A}^H)^{-1}$

Observation: The product of unitary matrices is unitary:
If $\boldsymbol{A}^H = \boldsymbol{A}^{-1}$ and $\boldsymbol{B}^H = \boldsymbol{B}^{-1}$, then
$(\boldsymbol{AB})^H = \boldsymbol{B}^H \boldsymbol{A}^H = \boldsymbol{B}^{-1} \boldsymbol{A}^{-1} = (\boldsymbol{AB})^{-1}$.

Observation: Any unitary $\boldsymbol{A}$ satisfies: $\boldsymbol{A}^H \boldsymbol{A} = \boldsymbol{I}$.
I.e. if $\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^n$ are columns of $\boldsymbol{A}$,
then $(\boldsymbol{x}^i)^H \boldsymbol{x}^j = 0$ for $i \neq j$ and $(\boldsymbol{x}^i)^H \boldsymbol{x}^i = 1$.

$$\boldsymbol{A} = \left[\begin{array}{c|c|c} | & & | \\ \boldsymbol{x}^1 & \ldots & \boldsymbol{x}^n \\ | & & | \end{array}\right]$$

Fact: Any $\boldsymbol{x} \in \mathbb{C}^n$ such that $\boldsymbol{x}^H \boldsymbol{x} = 1$
can be completed to a unitary matrix.

$$\left[\begin{array}{c|c} | & \\ \boldsymbol{x} & \\ | & \end{array}\right]$$

Diagonalization of Hermitian matrices

Theorem: Every Hermitian matrix $\boldsymbol{A}$ has all eigenvalues real. Also, a unitary matrix $\boldsymbol{R}$ exists, such that $\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}$ is diagonal.

Example: Diagonalize a Hermitian matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$.

$$p_{\boldsymbol{A}}(t) = \begin{vmatrix} 1-t & 1+i \\ 1-i & 2-t \end{vmatrix} = (1-t)(2-t) - (1-i)(1+i) = t^2 - 3t$$

Eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 3$ and $\lambda_2 = 0$.

The corresponding unitary matrix composed from eigenvectors is: $\boldsymbol{R} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$

$\boldsymbol{R}^{-1} = \boldsymbol{R}^{H} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$. (Indeed $\boldsymbol{R}$ is self-inverse: $\boldsymbol{R}^{-1} = \boldsymbol{R}$.)

The diagonalization goes by the product: $\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$.

If we revert the order of the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 3$, then we get:

$\boldsymbol{S} = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}$, $\boldsymbol{S}^{-1} = \boldsymbol{S}^{H} = \begin{pmatrix} \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \end{pmatrix}$ and $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$.

Proof

By induction on n, the theorem holds for $n = 1$. Denote $\boldsymbol{A}_n = \boldsymbol{A}$.

In $\mathbb{C}$, the matrix $\boldsymbol{A}_n$ has an eigenvalue λ with an eigenvector $\boldsymbol{x}$.

Scale $\boldsymbol{x}$ by the factor $\frac{1}{\sqrt{\boldsymbol{x}^H \boldsymbol{x}}}$, to get $\boldsymbol{x}$ satisfying $\boldsymbol{x}^H \boldsymbol{x} = 1$.

Extend (by the fact above) $\boldsymbol{x}$ to a unitary matrix $\boldsymbol{P}_n$.

$\boldsymbol{P}_n^H \boldsymbol{A}_n \boldsymbol{P}_n$ is Hermitian: $(\boldsymbol{P}_n^H \boldsymbol{A}_n \boldsymbol{P}_n)^H = \boldsymbol{P}_n^H \boldsymbol{A}_n^H (\boldsymbol{P}_n^H)^H = \boldsymbol{P}_n^H \boldsymbol{A}_n \boldsymbol{P}_n$.

Since $\boldsymbol{A}_n \boldsymbol{x} = \lambda \boldsymbol{x}$, the matrix $\boldsymbol{A}_n \boldsymbol{P}_n$ has $\lambda \boldsymbol{x}$ as the first column.

As $\boldsymbol{P}_n$ is unitary, the first column of $\boldsymbol{P}_n^H \boldsymbol{A}_n \boldsymbol{P}_n$ is $\boldsymbol{P}_n^H \boldsymbol{A}_n \boldsymbol{x} =$
$\boldsymbol{P}_n^H (\boldsymbol{A}_n \boldsymbol{x}) = \boldsymbol{P}_n^H (\lambda \boldsymbol{x}) = \lambda \boldsymbol{P}_n^H \boldsymbol{x} = \lambda (1, 0, \ldots, 0)^T = (\lambda, 0, \ldots, 0)^T$.

As $\boldsymbol{P}_n^H \boldsymbol{A}_n \boldsymbol{P}_n$ is Hermitian, $\lambda \in \mathbb{R}$ and the rest of the first row is $\mathbf{0}^T$.

Hence $\boldsymbol{P}_n^H \boldsymbol{A}_n \boldsymbol{P}_n = \begin{array}{|c|c|} \hline \lambda & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{A}_{n-1} \\ \hline \end{array}$, where $\boldsymbol{A}_{n-1}$ is Hermitian.

By the induction hypothesis, $\boldsymbol{R}_{n-1}^{-1} \boldsymbol{A}_{n-1} \boldsymbol{R}_{n-1} = \boldsymbol{D}_{n-1}$
for some unitary matrix $\boldsymbol{R}_{n-1}$ and a diagonal matrix $\boldsymbol{D}_{n-1}$.

Choose $\boldsymbol{R}_n = \boldsymbol{P}_n \cdot \begin{array}{|c|c|} \hline 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1} \\ \hline \end{array}$, products of unitary matrices are unitary. Now:

$$\boldsymbol{R}_n^{-1}\boldsymbol{A}_n\boldsymbol{R}_n = \boldsymbol{R}_n^H\boldsymbol{A}_n\boldsymbol{R}_n = \begin{array}{|c|c|} \hline 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1}^H \\ \hline \end{array} \cdot \boldsymbol{P}_n^H\boldsymbol{A}_n\boldsymbol{P}_n \cdot \begin{array}{|c|c|} \hline 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1} \\ \hline \end{array} =$$

$$= \begin{array}{|c|c|} \hline 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1}^H \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \lambda & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{A}_{n-1} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline 1 & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_{n-1} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \lambda & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{D}_{n-1} \\ \hline \end{array} = \boldsymbol{D}_n$$

Theorem: Every *real symmetric* matrix $\boldsymbol{A}$ has all eigenvalues real. Also, an *orthogonal* matrix $\boldsymbol{R}$ exists, such that $\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}$ is diagonal.

By the same proof, only the eigenvector $\boldsymbol{x}$ shall be real. Such $\boldsymbol{x}$ exists, since the system $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$ has all coefficients real.

Example

Given $\boldsymbol{A} = \boldsymbol{A}_3 =$

$$= \begin{pmatrix} 2 & \frac{2(1+i)}{3} & \frac{-1-i}{3} \\ \frac{2(1-i)}{3} & \frac{2}{3} & \frac{2i}{3} \\ \frac{-1+i}{3} & -\frac{2i}{3} & \frac{7}{3} \end{pmatrix}$$

$p_{\boldsymbol{A}_3}(t) = t^3 - 5t^2 + 6t$,

$\lambda = 2$ corresponds to $\left(1, \frac{1}{2}, 1\right)^T$,

we scale it to $\boldsymbol{x} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)^T$.

We extend $\boldsymbol{x}$ to unitary $\boldsymbol{P}_3 = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and get Hermitian

$$\boldsymbol{P}_3^H \boldsymbol{A}_3 \boldsymbol{P}_3 = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 1 & 1+i \\ 0 & 1-i & 2 \end{array}\right)$$

By induction hypothesis we diagonalize $\boldsymbol{R}_2^{-1} \boldsymbol{A}_2 \boldsymbol{R}_2 = \boldsymbol{D}_2$:

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1+i}{\sqrt{3}} \\ \frac{1-i}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

For

$$\boldsymbol{R}_3 = \boldsymbol{P}_3 \cdot \begin{array}{|c|c|} \hline \mathbf{1} & \mathbf{0}^T \\ \hline \mathbf{0} & \boldsymbol{R}_2 \\ \hline \end{array} = \begin{pmatrix} \frac{2}{3} & \frac{-3+i}{3\sqrt{3}} & \frac{-1-2i}{3\sqrt{3}} \\ \frac{1}{3} & \frac{2i}{3\sqrt{3}} & \frac{4+2i}{3\sqrt{3}} \\ \frac{2}{3} & \frac{3-2i}{3\sqrt{3}} & \frac{-1+i}{3\sqrt{3}} \end{pmatrix}$$

then holds: $\boldsymbol{R}_3^{-1} \boldsymbol{A}_3 \boldsymbol{R}_3 =$

$$= \boldsymbol{D}_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$