# Similar matrices

The matrix of a linear map $f$ on $V$ is not unique, since it depends on the basis. Matrices of the same map, but w.r.t. different bases shall have the same eigenvalues.

$$[f]_{XX} = [id]_{YX}[f]_{YY}[id]_{XY}$$

$$\begin{aligned}
[f(u)]_X &= [f]_{XX}[u]_X \\
&= [id]_{YX}[f(u)]_Y = [id]_{YX}[f]_{YY}[u]_Y \\
&= [id]_{YX}[f]_{YY}[id]_{XY}[u]_X
\end{aligned}$$

Note that $[id]_{YX} = [id]_{XY}^{-1}$

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Definition Matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{K}^{n \times n}$ are *similar* if there exists a regular matrix $\boldsymbol{R}$ such that $\boldsymbol{A} = \boldsymbol{R}^{-1}\boldsymbol{B}\boldsymbol{R}$.

Observation: If $\boldsymbol{A}$ is similar to $\boldsymbol{B}$, i.e. $\boldsymbol{B} = \boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^{-1}$,
and an eigenvalue $\lambda$ corresponds to an eigenvector $\boldsymbol{x}$ in $\boldsymbol{A}$,
then $\lambda$ is also an eigenvalue of $\boldsymbol{B}$ and corresponds here to $\boldsymbol{R}\boldsymbol{x}$.

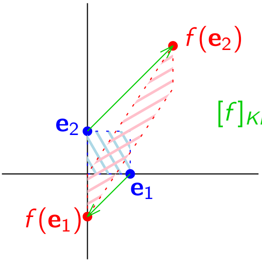
Proof: For $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{x}$ holds: $\boldsymbol{B}\boldsymbol{y} = \boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{x} = \boldsymbol{R}\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{R}\boldsymbol{x} = \lambda\boldsymbol{y}$.

Observation: If $\boldsymbol{B} = \boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^{-1}$ then $p_{\boldsymbol{B}}(t) = p_{\boldsymbol{A}}(t)$.

Proof: $p_{\boldsymbol{B}}(t) = \det(\boldsymbol{B} - t\boldsymbol{I}) = \det(\boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^{-1} - \boldsymbol{R}(t\boldsymbol{I})\boldsymbol{R}^{-1}) =$
$\det(\boldsymbol{R}(\boldsymbol{A} - t\boldsymbol{I})\boldsymbol{R}^{-1}) = \det(\boldsymbol{R})\det(\boldsymbol{A} - t\boldsymbol{I})\det(\boldsymbol{R}^{-1}) = p_{\boldsymbol{A}}(t)$

# Example — a linear map in the plane

Does the following linear map have a better description?




$$[f]_{KK} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

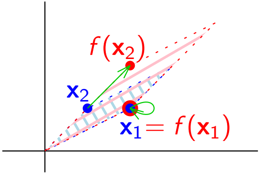
Characteristic polynomial:

$$p_{[f]_{KK}}(t) = \begin{vmatrix} -t & 2 \\ -1 & 3-t \end{vmatrix} = t^2 - 3t + 2 = (t-1)(t-2)$$

The eigenvalue $\lambda_1 = 1$ has eigenvector $\boldsymbol{x}_1 = (2,1)^T$, and the eigenvalue $\lambda_2 = 2$ has eigenvector $\boldsymbol{x}_2 = (1,1)^T$.

With respect to the new basis $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2\} = \{(2,1)^T, (1,1)^T\}$ the matrix of *the same* linear map $f$ is *diagonal*:

$$[f]_{XX} = [id]_{KX}[f]_{KK}[id]_{XK} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$




$$[f]_{XX} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Less formally: the plane is fixed along the line through $\boldsymbol{x}_1$ and twice stretched along the line through $\boldsymbol{x}_2$.

Observe that the eigenvalues and eigenvectors are preserved.

# Algebraic and geometric multiplicity

Observation: If a basis $X$ contains an eigenvector $\boldsymbol{x}$ of $f$, then the coordinate corresponding to $\boldsymbol{x}$ is scaled by $\lambda$ under $f$.
In matrix terms: $[f]_{XX}$ contains in the column corresponding to $\boldsymbol{x}$ only $\lambda$ at the diagonal and otherwise zeroes.

Proof: When an eigenvector $\boldsymbol{u}$ is the $i$-th vector of a basis $X$, then the $i$-th column of $[f]_{XX}$ is $[f(\boldsymbol{u})]_X = [\lambda \boldsymbol{u}]_X = \lambda [\boldsymbol{u}]_X = \lambda \boldsymbol{e}^i$.

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Theorem: The geometric multiplicity of an eigenvalue $\lambda$ of a matrix $\boldsymbol{A}$ is smaller or equal to its algebraic multiplicity.

Proof: View $\boldsymbol{A} \in \mathbb{K}^{n \times n}$ as the matrix of a linear map $f : \mathbb{K}^n \to \mathbb{K}^n$ w.r.t. the standard basis $K$, i.e. $\boldsymbol{A} = [f]_{K,K}$.
Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ be a basis of the space of eigenvectors of $\lambda$, i.e. $k$ is its geometric multiplicity.

Extend this basis to a basis $X$ of $\mathbb{K}^n$.
Then $[f]_{X,X} = [id]_{X,K}^{-1} \boldsymbol{A} [id]_{X,K}$ is similar to $\boldsymbol{A}$. Also $[f]_{X,X}$ has on the first $k$ columns $\lambda$ at the diagonal and otherwise zeroes.

Hence $(\lambda - t)^k$ divides $p_{[f]_{X,X}}(t)$. Since $\boldsymbol{A}$ and $[f]_{X,X}$ have equal characteristic polynomials, $\lambda$ has algebraic multiplicity at least $k$.

# Example

$\boldsymbol{A} = \begin{pmatrix} -1 & 7 & -5 \\ -2 & 7 & -4 \\ -1 & 3 & -1 \end{pmatrix}$ $p_{\boldsymbol{A}}(t) = -t^3 + 5t^2 - 8x + 4 = (t-2)^2(t-1)$ eigenvalues are: $2$ of *algebraic multiplicity* $2$ and $1$ of algebraic multiplicity $1$.

$$\boldsymbol{A} - 2\boldsymbol{I}_3 = \begin{pmatrix} -3 & 7 & -5 \\ -2 & 5 & -4 \\ -1 & 3 & -3 \end{pmatrix} \sim\sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenvalue $2$ has in $\boldsymbol{A}$ *geometric multiplicity* only $1$.

We extend the eigenvector $(3,2,1)^T$ for $2$ to a basis $X$, e.g. $X = \{(3,2,1)^T, (2,2,1)^T, (1,1,1)^T\}$.

The matrix $\boldsymbol{A}$ is similar to $[id]_{X,K}^{-1}\boldsymbol{A}[id]_{X,K} =$

$$= \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 7 & -5 \\ -2 & 7 & -4 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Example

$$\boldsymbol{A} = \begin{pmatrix} -1 & 7 & -5 \\ -2 & 7 & -4 \\ -1 & 3 & -1 \end{pmatrix}$$

$p_{\boldsymbol{A}}(t) = -t^3 + 5t^2 - 8x + 4 = (t-2)^2(t-1)$
eigenvalues are: 2 of *algebraic multiplicity* 2 and 1 of algebraic multiplicity 1.

$$\boldsymbol{A} - 2\boldsymbol{I}_3 = \begin{pmatrix} -3 & 7 & -5 \\ -2 & 5 & -4 \\ -1 & 3 & -3 \end{pmatrix} \sim\sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenvalue 2 has in $\boldsymbol{A}$ *geometric multiplicity* only 1.

Compare with,

$$\boldsymbol{B} = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

has the same characteristic polynomial
$p_{\boldsymbol{B}}(t) = -t^3 + 5t^2 - 8x + 4 = (t-2)^2(t-1)$
and the same eigenvalues, i.e. 2 of algebraic multiplicity 2 and 1 of algebraic multiplicity 1.

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the eigenvalue 2 has in $\boldsymbol{B}$ geometric multiplicity 2.

# Example

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$p_{\boldsymbol{B}}(t) = -t^3 + 5t^2 - 8x + 4 = (t-2)^2(t-1)$
and the same eigenvalues, i.e. $2$ of algebraic multiplicity $2$ and $1$ of algebraic multiplicity $1$.

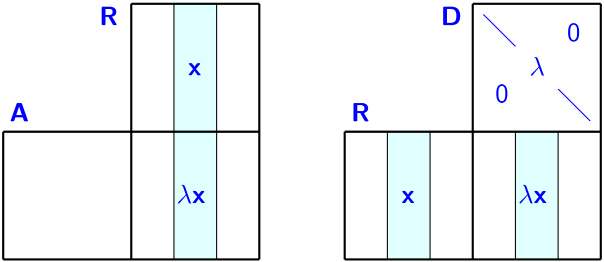
W.r.t. (by coincidence the same) basis $X$ we get $[id]_{X,K}^{-1}\boldsymbol{B}[id]_{X,K} =$

$$= \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Diagonalization

Observation: A matrix $\boldsymbol{A} \in \mathbb{K}^{n \times n}$ is similar to a diagonal matrix if and only if $\mathbb{K}^n$ has a basis consisting of eigenvectors of $\boldsymbol{A}$.

Proof: $\boldsymbol{AR} = \boldsymbol{RD}$ with diagonal matrix $\boldsymbol{D}$, iff for every $i$ there exists a vector $\boldsymbol{x}$ (the $i$-th column of $\boldsymbol{R}$) such that $\boldsymbol{Ax} = \lambda \boldsymbol{x} = d_{ii} \boldsymbol{x}$.




$$\mathbf{A} = \mathbf{RDR}^{-1} \iff \mathbf{AR} = \mathbf{RD} \iff \mathbf{R}^{-1}\mathbf{AR} = \mathbf{D}$$

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Definition: A matrix similar to a diagonal matrix is *diagonalizable*.

Corollary: If a square matrix of order $n$ has $n$ distinct eigenvalues, then it is diagonalizable.

Corollary: When $p_{\boldsymbol{A}}(t) = \prod_i (t - \lambda_i)^{r_i}$, then:

$$\boldsymbol{A} \text{ is diagonalizable} \quad \Longleftrightarrow \quad \dim(\mathrm{Ker}(\boldsymbol{A} - \lambda_i \boldsymbol{I})) = r_i$$

Corollary: If $\boldsymbol{A} = \boldsymbol{R}^{-1}\boldsymbol{DR}$, then for any $k : \boldsymbol{A}^k = \boldsymbol{R}^{-1}\boldsymbol{D}^k\boldsymbol{R}$.
$\boldsymbol{A}^k = (\boldsymbol{R}^{-1}\boldsymbol{DR})^k = \boldsymbol{R}^{-1}\boldsymbol{D}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{D}\boldsymbol{R}\boldsymbol{R}^{-1} \cdots \boldsymbol{R}^{-1}\boldsymbol{DR} = \boldsymbol{R}^{-1}\boldsymbol{D}^k\boldsymbol{R}$.