Solving a homogeneous system of first order linear differential equations with constant coefficients

$$\begin{array}{ccccccc} y_1' & = & a_{1,1}y_1 & + & \dots & + & a_{1,n}y_n \\ \vdots & & \vdots & & & & \vdots \\ y_n' & = & a_{n,1}y_1 & + & \dots & + & a_{n,n}y_n \end{array} \quad \text{yield } \boldsymbol{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix}$$

For an eigenvalue λ and an eigenvector $(k_1, \dots, k_n)^T$ of $\boldsymbol{A}$, the n-tuple of functions $y_i(x) := k_i e^{\lambda x}$ solves the original system:

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} (k_1 e^{\lambda x})' \\ \vdots \\ (k_n e^{\lambda x})' \end{pmatrix} = \begin{pmatrix} \lambda k_1 e^{\lambda x} \\ \vdots \\ \lambda k_n e^{\lambda x} \end{pmatrix} = e^{\lambda x} \lambda \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = e^{\lambda x} \boldsymbol{A} \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} =$$

$$= \begin{pmatrix} a_{1,1}k_1 e^{\lambda x} & + \dots + & a_{1,n}k_n e^{\lambda x} \\ \vdots & & \vdots \\ a_{n,1}k_1 e^{\lambda x} & + \dots + & a_{n,n}k_n e^{\lambda x} \end{pmatrix} = \begin{pmatrix} a_{1,1}y_1 & + \dots + & a_{1,n}y_n \\ \vdots & & \vdots \\ a_{n,1}y_1 & + \dots + & a_{n,n}y_n \end{pmatrix}$$

Note that other eigenvalues yield another sets of particular solutions.

The overall solution is any linear combination of particular solutions that satisfies all boundary conditions.