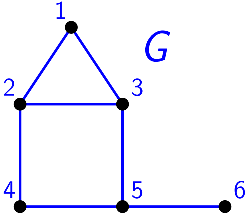
Spanning tree of a graph

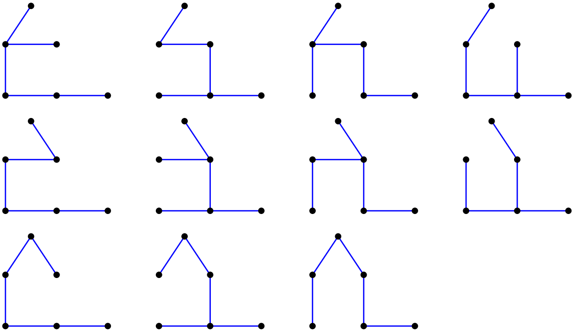
Definition: A *spanning tree* of a connected graph G is its subgraph that is a tree and that contains all vertices of G.

The number of spanning trees of G is denoted by $\kappa(G)$.

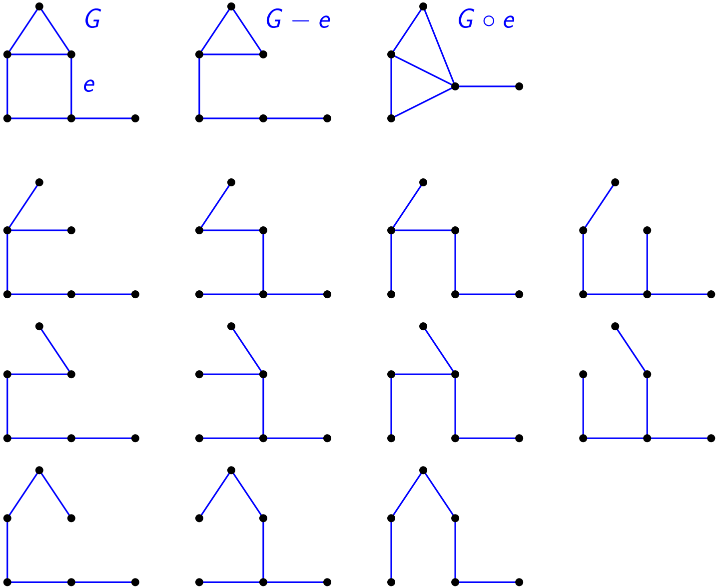
Example:



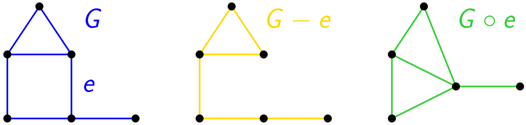
$\kappa(G) = 11$



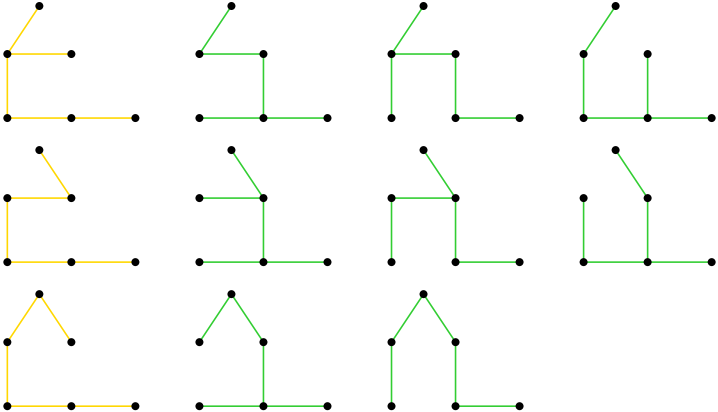
Recurrence



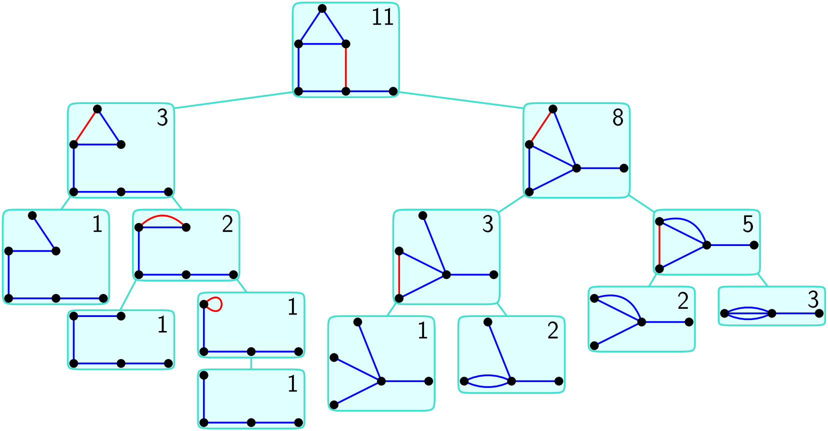
Recurrence



$$\kappa(G) = \kappa(G - e) + \kappa(G \circ e)$$



Part of the recurrence tree



The recurrence tree may have exponentially many leaves.

Determinants — the number of spanning trees of a graph

Definition: The *Laplace matrix* of a graph G on $V_G = \{v_1, \ldots, v_n\}$ is $\boldsymbol{L}_G \in \mathbb{R}^{n \times n}$ such that:

$$(\boldsymbol{L}_G)_{i,j} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } (v_i, v_j) \in E_G \\ 0 & \text{otherwise} \end{cases}$$

Observation: The Laplace matrix $\boldsymbol{L}_G$ is singular.

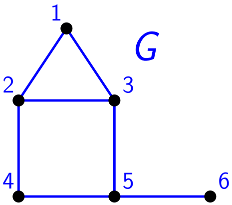
Observation: If G is not connected then $\boldsymbol{L}_G^{1,1}$ is singular.

Theorem: Any graph G on at least two vertices has $\det(\boldsymbol{L}_G^{1,1})$ distinct spanning trees.

The theorem holds also for multigraphs when:

- Trees have no loops (as they are cycles of length 1).
- Distinct threads of a multiedge yield different trees.
- $-(\boldsymbol{L}_G)_{i,j}$ for $i \neq j$ is the multiplicity of the edge (v_i, v_j).
- $\deg(v_i)$ counts edges with multiplicities, but not loops.

Determinants — the number of spanning trees of a graph



Laplace matrix

$$\boldsymbol{L}_G = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\det(\boldsymbol{L}_G^{1,1}) = \begin{vmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = 11$$

Properties of $\det(\boldsymbol{L}^{i,j})$

$$\det(\boldsymbol{L}_G^{1,1}) = \begin{array}{c} \begin{vmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} \end{array} \begin{array}{l} \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \\ \mathrm{V} \\ \mathrm{VI} \end{array}$$

$$\boldsymbol{L}_G = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} \begin{array}{c} \mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI} \\ \mathrm{III} \\ \mathrm{IV} \\ \mathrm{V} \\ \mathrm{VI} \end{array}$$

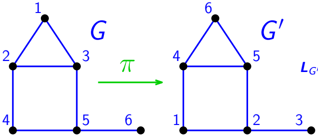
$$= (-1) \cdot \begin{vmatrix} -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} \begin{array}{l} \mathrm{I} \\ \mathrm{III} \\ \mathrm{IV} \\ \mathrm{V} \\ \mathrm{VI} \end{array} = -\det(\boldsymbol{L}_G^{2,1})$$

Analogously with column operations we get: $\det(\boldsymbol{L}_G^{2,1}) = -\det(\boldsymbol{L}_G^{2,2})$.

Corolary: For any $i, j \in \{1, \ldots, n\} : \det(\boldsymbol{L}_G^{i,j}) = (-1)^{i+j} \det(\boldsymbol{L}_G^{1,1})$.

Laplace matrices of isomorphic graphs

$$\boldsymbol{L}_G = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$



$$\boldsymbol{L}_{G'} = \begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

$$\det(\boldsymbol{L}_G^{1,1}) = \det(\boldsymbol{L}_{G'}^{6,6}) = \det(\boldsymbol{L}_{G'}^{1,1})$$

$$\begin{vmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 \\ 0 & -1 & 0 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & -1 \\ -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{vmatrix}$$

Matrices $\boldsymbol{L}_G^{1,1}$, $\boldsymbol{L}_{G'}^{6,6}$ differ only by the permutation of *rows and columns* by π. This π is applied twice: on rows and on columns. Even if $\mathrm{sgn}(\pi) = -1$ the overall determinant sign does not change.

Proof

Theorem: If a multigraph G has $|V_G| \geq 2$ then $\kappa(G) = \det(\boldsymbol{L}_G^{1,1})$.

Proof: W.l.o.g. G is connected. By induction on $m = |E_G|$.

Induction basis: For $m = 1$ the graph G has only two vertices and $\kappa(G) = 1 = \deg(v_2) = (\boldsymbol{L}_G)_{2,2} = \det(\boldsymbol{L}_G^{1,1})$.

Induction step: Choose any $e \in E_G$, w.l.o.g. $e = (v_1, v_2)$.

Denote $\boldsymbol{A} = \boldsymbol{L}_G^{1,1}$, $\boldsymbol{B} = \boldsymbol{L}_{G-e}^{1,1}$ and $\boldsymbol{C} = \boldsymbol{L}_{G \circ e}^{1,1} = \boldsymbol{A}^{1,1} = \boldsymbol{B}^{1,1}$

... $\boldsymbol{C}$ is the submatrix of $\boldsymbol{L}_G$ corresponding to $v_3, \ldots, v_n$.

By induction hypothesis $\kappa(G - e) = \det(\boldsymbol{B})$, $\kappa(G \circ e) = \det(\boldsymbol{C})$.

Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are identical except for $b_{1,1} = a_{1,1} - 1$, since by the deletion of e the degree of v_2 drops by 1 . We express the first column of $\boldsymbol{A}$ as the sum of the first column of $\boldsymbol{B}$ and the vector $\boldsymbol{e}^1$.

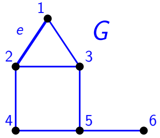
By the linearity of $\det(\boldsymbol{A})$ along this split of its first column we obtain $\det(\boldsymbol{A}) = \det(\boldsymbol{B}) + \det(\boldsymbol{C})$. Now we conclude:

$\kappa(G) = \kappa(G - e) + \kappa(G \circ e) = \det(\boldsymbol{L}_{G-e}^{1,1}) + \det(\boldsymbol{L}_{G \circ e}^{1,1}) = \det(\boldsymbol{L}_G^{1,1})$

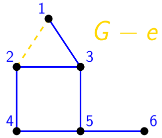
Example

$$\kappa(G) = \kappa(G-e) + \kappa(G \circ e) = \det(\boldsymbol{L}^{1,1}_{G-e}) + \det(\boldsymbol{L}^{1,1}_{G\circ e}) \overset{??}{=} \det(\boldsymbol{L}^{1,1}_{G})$$

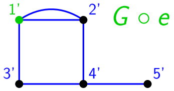
$$\boldsymbol{L}_G = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$



$$\boldsymbol{L}_{G-e} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$



$$\boldsymbol{L}_{G\circ e} = \begin{pmatrix} 3 & -2 & -1 & 0 & 0 \\ -2 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$



Example

$$\kappa(G) = \kappa(G-e) + \kappa(G \circ e) = \det(L_{G-e}^{1,1}) + \det(L_{G\circ e}^{1,1}) \overset{??}{=} \det(L_G^{1,1})$$

$$L_G = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \qquad L_G^{1,1} = A$$

$$L_{G-e} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \qquad L_{G-e}^{1,1} = B$$

$$L_{G\circ e} = \begin{pmatrix} 3 & -2 & -1 & 0 & 0 \\ -2 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \qquad L_{G\circ e}^{1,1} = C$$

Example

$$\kappa(G) = \kappa(G-e) + \kappa(G \circ e) = \det(\boldsymbol{L}_{G-e}^{1,1}) + \det(\boldsymbol{L}_{G\circ e}^{1,1}) = \det(\boldsymbol{L}_{G}^{1,1})$$

$$\det(\boldsymbol{A}) = \begin{vmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2+1 & -1 & -1 & 0 & 0 \\ -1+0 & 3 & 0 & -1 & 0 \\ -1+0 & 0 & 2 & -1 & 0 \\ 0+0 & -1 & -1 & 3 & -1 \\ 0+0 & 0 & 0 & -1 & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix} = \det(\boldsymbol{B}) + \det(\boldsymbol{C})$$

Spanning trees of complete graphs — Cayley's formula

Theorem: The complete graph K_n has n^{n-2} spanning trees.

Proof:

$$\kappa(K_n) = \det(\boldsymbol{L}_{K_n}^{1,1}) = \begin{vmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \dots & -1 & n-1 \end{vmatrix} =$$

$$= \begin{vmatrix} n-1 & -1 & -1 & \dots & -1 \\ -n & n & 0 & \dots & 0 \\ -n & 0 & n & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -n & 0 & \dots & 0 & n \end{vmatrix} = \begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 0 & n & 0 & \dots & 0 \\ 0 & 0 & n & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & n \end{vmatrix} = n^{n-2}$$