Solving a system $\boldsymbol{Ax} = \boldsymbol{b}$ with regular $\boldsymbol{A}$

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 &= b_2
\end{aligned}
$$

From the 2nd express $x_1 = \frac{b_2 - a_{2,2}x_2}{a_{2,1}}$ and substitute to the 1st:

$$
\begin{aligned}
a_{1,1}\frac{b_2 - a_{2,2}x_2}{a_{2,1}} + a_{1,2}x_2 &= b_1 \quad \Leftrightarrow \\
\frac{a_{1,1}b_2 - a_{1,1}a_{2,2}x_2 + a_{1,2}a_{2,1}x_2}{a_{2,1}} &= b_1 \quad \Leftrightarrow \\
(-a_{1,1}a_{2,2} + a_{1,2}a_{2,1})x_2 &= a_{2,1}b_1 - a_{1,1}b_2 \quad \Leftrightarrow \\
x_2 &= \frac{a_{1,1}b_2 - a_{2,1}b_1}{a_{1,1}a_{2,2} - a_{2,1}a_{1,2}}
\end{aligned}
$$

$$
x_1 = \frac{b_2 - a_{2,2}\frac{a_{1,1}b_2 - a_{2,1}b_1}{a_{1,1}a_{2,2} - a_{2,1}a_{1,2}}}{a_{2,1}} = \cdots = \frac{b_1a_{2,2} - b_2a_{1,2}}{a_{1,1}a_{2,2} - a_{2,1}a_{1,2}}
$$

For three equations

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 &= b_2 \\
a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 &= b_3
\end{aligned}
$$

In an analogous way:

(express an unknown from one equation and substitute it to the others)

$$x_1 = \frac{b_1 a_{2,2} a_{3,3} + a_{1,2} a_{2,3} b_3 + a_{1,3} b_2 a_{3,2} - b_1 a_{2,3} a_{3,2} - a_{1,2} b_2 a_{3,3} - a_{1,3} a_{2,2} b_3}{a_{1,1} a_{2,2} a_{3,3} + a_{1,2} a_{2,3} a_{3,1} + a_{1,3} a_{2,1} a_{3,2} - a_{1,1} a_{2,3} a_{3,2} - a_{1,2} a_{2,1} a_{3,3} - a_{1,3} a_{2,2} a_{3,1}}$$

$$x_2 = \frac{a_{1,1} b_2 a_{3,3} + b_1 a_{2,3} a_{3,1} + a_{1,3} a_{2,1} b_3 - a_{1,1} a_{2,3} b_3 - b_1 a_{2,1} a_{3,3} - a_{1,3} b_2 a_{3,1}}{a_{1,1} a_{2,2} a_{3,3} + a_{1,2} a_{2,3} a_{3,1} + a_{1,3} a_{2,1} a_{3,2} - a_{1,1} a_{2,3} a_{3,2} - a_{1,2} a_{2,1} a_{3,3} - a_{1,3} a_{2,2} a_{3,1}}$$

$$x_3 = \frac{a_{1,1} a_{2,2} b_3 + a_{1,2} b_2 a_{3,1} + b_1 a_{2,1} a_{3,2} - a_{1,1} b_2 a_{3,2} - a_{1,2} a_{2,1} b_3 - b_1 a_{2,2} a_{3,1}}{a_{1,1} a_{2,2} a_{3,3} + a_{1,2} a_{2,3} a_{3,1} + a_{1,3} a_{2,1} a_{3,2} - a_{1,1} a_{2,3} a_{3,2} - a_{1,2} a_{2,1} a_{3,3} - a_{1,3} a_{2,2} a_{3,1}}$$

Determinants

Review: S_n the group of permutations over the set $\{1, \ldots, n\}$.
The sign of $p \in S_n$ is $\text{sgn}(p) = (-1)^{\#\text{ of inversions of } p}$

Definition: The *determinant* of a matrix $\boldsymbol{A} \in \mathbb{K}^{n \times n}$ is

$$\det(\boldsymbol{A}) = \sum_{p \in S_n} \text{sgn}(p) \prod_{i=1}^{n} a_{i,p(i)}$$

Denoted also by $|\boldsymbol{A}|$.

Example: For $\boldsymbol{A} \in \mathbb{K}^{2 \times 2}$ we have $S_2 = \{(1,2), (2,1)\}$.

for $p = (1,2)$ we get $\text{sgn}(p) = +1$ and $\prod_{i=1}^{n} a_{i,p(i)} = a_{1,1} a_{2,2}$
for $p = (2,1)$ we get $\text{sgn}(p) = -1$ and $\prod_{i=1}^{n} a_{i,p(i)} = a_{1,2} a_{2,1}$

Hence:

$$\det(\boldsymbol{A}) = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = (+1) \cdot a_{1,1} a_{2,2} + (-1) \cdot a_{1,2} a_{2,1}$$

Intuitively:

$$+ \begin{pmatrix} a_{1,1} & \cdot \\ \cdot & a_{2,2} \end{pmatrix} - \begin{pmatrix} \cdot & a_{1,2} \\ a_{2,1} & \cdot \end{pmatrix}$$

For matrices of order three we have six possible permutations

$S_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$

permutations $p = (1,2,3),(2,3,1)$ and $(3,1,2)$ have $sgn(p) = +1$

permutations $p = (1,3,2),(2,1,3)$ and $(3,2,1)$ have $sgn(p) = -1$

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = \begin{array}{l} + a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1} \end{array}$$

$$+ \begin{pmatrix} a_{1,1} & \cdot & \cdot \\ \cdot & a_{2,2} & \cdot \\ \cdot & \cdot & a_{3,3} \end{pmatrix} + \begin{pmatrix} \cdot & a_{1,2} & \cdot \\ \cdot & \cdot & a_{2,3} \\ a_{3,1} & \cdot & \cdot \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & a_{1,3} \\ a_{2,1} & \cdot & \cdot \\ \cdot & a_{3,2} & \cdot \end{pmatrix}$$

$$- \begin{pmatrix} a_{1,1} & \cdot & \cdot \\ \cdot & \cdot & a_{2,3} \\ \cdot & a_{3,2} & \cdot \end{pmatrix} - \begin{pmatrix} \cdot & a_{1,2} & \cdot \\ a_{2,1} & \cdot & \cdot \\ \cdot & \cdot & a_{3,3} \end{pmatrix} - \begin{pmatrix} \cdot & \cdot & a_{1,3} \\ \cdot & a_{2,2} & \cdot \\ a_{3,1} & \cdot & \cdot \end{pmatrix}$$

Sarrus rule:

$$\begin{array}{c} + \\ \\ - \end{array} \left| \begin{array}{ccc} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{array} \right| \begin{array}{cc} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{array}$$

Only for matrices 3×3 *!!!*

Observation: If $\boldsymbol{A}$ has a zero row, then $\det(\boldsymbol{A}) = 0$.

Proof: Every product $\prod_{i=1}^{n} a_{i,p(i)}$ contains a term from the zero row.

Observation: For triangular (also for diagonal) matrices we get:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{n,n} \end{vmatrix} = a_{1,1}a_{2,2}\dots a_{n,n}$$

Proof: Every permutation p, for which there is an index with $i < p(i)$ must have an index $j > p(j)$. Consequently $a_{j,p(j)} = 0$.

If for a contradiction was $i \leq p(i)$ for all $i \in \{1, \dots, n\}$ and for some i_1 was $i_1 < p(i_1)$, then consider the sequence $i_1, i_2 = p(i_1), i_3 = p(i_2), \dots$ Since p is injective, this sequence is strictly increasing. Hence it is unbounded, a contradiction.

Only the product $a_{1,1}a_{2,2}\dots a_{n,n}$ corresponding to the identity permutation has no zero term from below the diagonal.

Properties of the determinant

Observation: $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$

Proof: For a $p \in S(n) : p(i) = j \Leftrightarrow p^{-1}(j) = i$

$$\det(\boldsymbol{A}^T) = \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} (\boldsymbol{A}^T)_{i,p(i)} = \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{p(i),i} =$$

$$= \sum_{p^{-1} \in S_n} \operatorname{sgn}(p^{-1}) \prod_{j=1}^{n} a_{j,p^{-1}(j)} = \det(\boldsymbol{A})$$

Properties of the determinant

Observation: $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$

Observation: (Rearranging columns according to a permutation q)
For $q \in S_n$ and $\boldsymbol{B} : b_{i,j} = a_{i,q(j)}$ holds $\det(\boldsymbol{B}) = \det(\boldsymbol{A}) \cdot \operatorname{sgn}(q)$.

Proof: $\det(\boldsymbol{B}) = \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} b_{i,p(i)} = \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{i,q(p(i))} =$

$$= \sum_{p \in S_n} \operatorname{sgn}(q) \operatorname{sgn}(q) \operatorname{sgn}(p) \prod_{i=1}^{n} a_{i,(q \circ p)(i)} =$$

$$= \operatorname{sgn}(q) \sum_{r \in S_n} \operatorname{sgn}(r) \prod_{i=1}^{n} a_{i,r(i)} = \operatorname{sgn}(q) \det(\boldsymbol{A})$$

for $r = q \circ p$; note that $p \to r$ is a bijection on S_n

Properties of the determinant

Observation: $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$

Observation: (Rearranging columns according to a permutation q)
For $q \in S_n$ and $\boldsymbol{B} : b_{i,j} = a_{i,q(j)}$ holds $\det(\boldsymbol{B}) = \det(\boldsymbol{A}) \cdot \operatorname{sgn}(q)$.

Corollaries:

- The same holds for any rearrangement of rows.
- Exchange of two rows/columns changes the sign of the determinant.
- For fields $\text{char} \neq 2$: If a matrix $\boldsymbol{A}$ has two rows/columns identical, then $\det(\boldsymbol{A}) = 0$. Proof: $\alpha = -\alpha \Rightarrow \alpha = 0$.

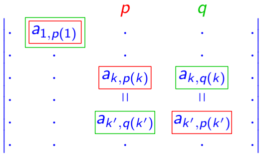
Properties of the determinant

Observation: $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$

Observation: (Rearranging columns according to a permutation q)
For $q \in S_n$ and $\boldsymbol{B} : b_{i,j} = a_{i,q(j)}$ holds $\det(\boldsymbol{B}) = \det(\boldsymbol{A}) \cdot \operatorname{sgn}(q)$.

Lemma: If $\boldsymbol{A}$ has two rows/columns identical, then $\det(\boldsymbol{A}) = 0$.

Proof: Let the k-th row match the k'-th.
Then any $p \in S_n$ and $q = (k, k') \circ p$ yield:
$\prod_{i=1}^{n} a_{i,p(i)} = \prod_{i=1}^{n} a_{i,q(i)}$, but $\operatorname{sgn}(p) = -\operatorname{sgn}(q)$.



As $p \leftrightarrow q$ is a bijection between permutations with opposite signs, the terms in $\det(\boldsymbol{A})$ can therefore be paired to cancel each other.

Linearity of the determinant

Theorem: The determinant of a matrix is linearly dependent on each its row and column, i.e. w.r.t. the scalar multiple of a row:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ t \cdot a_{i,1} & t \cdot a_{i,2} & \dots & t \cdot a_{i,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} = t \cdot \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{i,1} & a_{i,2} & \dots & a_{i,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix}$$

and w.r.t. the sum along a row:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ b_{i,1}+c_{i,1} & b_{i,2}+c_{i,2} & \dots & b_{i,n}+c_{i,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} = \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ b_{i,1} & b_{i,2} & \dots & b_{i,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} + \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ c_{i,1} & c_{i,2} & \dots & c_{i,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix}$$

Proof for the scalar multiple

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ t \cdot a_{i,1} & t \cdot a_{i,2} & \dots & t \cdot a_{i,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} = \sum_{p \in S_n} \operatorname{sgn}(p) \left(\left(\prod_{i=1}^{n} a_{i,p(i)} \right) \cdot t \right) =$$

$$= t \cdot \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{i=1}^{n} a_{i,p(i)} = t \cdot \begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{i,1} & a_{i,2} & \dots & a_{i,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix}$$

Proof for the addition

If matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ satisfy $a_{k,j} = \begin{cases} b_{i,j} + c_{i,j} & \text{when } k = i \text{ and} \\ b_{k,j} = c_{k,j} & \text{when } k \neq i \text{, then} \end{cases}$

$$
\begin{aligned}
\det(\boldsymbol{A}) &= \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{k=1}^{n} a_{k,p(k)} \\
&= \sum_{p \in S_n} a_{i,p(i)} \operatorname{sgn}(p) \prod_{k \in \{1,\dots,n\} \setminus i} a_{k,p(k)} \\
&= \sum_{p \in S_n} \left(b_{i,p(i)} + c_{i,p(i)} \right) \operatorname{sgn}(p) \prod_{k \in \{1,\dots,n\} \setminus i} a_{k,p(k)} \\
&= \sum_{p \in S_n} b_{i,p(i)} \operatorname{sgn}(p) \prod_{k \in \{1,\dots,n\} \setminus i} b_{k,p(k)} \\
&+ \sum_{p \in S_n} c_{i,p(i)} \operatorname{sgn}(p) \prod_{k \in \{1,\dots,n\} \setminus i} c_{k,p(k)} \\
&= \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{k=1}^{n} b_{k,p(k)} + \sum_{p \in S_n} \operatorname{sgn}(p) \prod_{k=1}^{n} c_{k,p(k)} \\
&= \det(\boldsymbol{B}) + \det(\boldsymbol{C})
\end{aligned}
$$

Example for the addition

$$\begin{vmatrix} b_{1,1}+c_{1,1} & b_{1,2}+c_{1,2} & b_{1,3}+c_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$$

$$\begin{aligned}
= \quad & (b_{1,1}+c_{1,1})a_{2,2}a_{3,3} + (b_{1,2}+c_{1,2})a_{2,3}a_{3,1} + (b_{1,3}+c_{1,3})a_{2,1}a_{3,2} \\
& -(b_{1,1}+c_{1,1})a_{2,3}a_{3,2} - (b_{1,2}+c_{1,2})a_{2,1}a_{3,3} - (b_{1,3}+c_{1,3})a_{2,2}a_{3,1} \\
\overset{(*)}{=} \quad & (b_{1,1}a_{2,2}a_{3,3} + b_{1,2}a_{2,3}a_{3,1} + b_{1,3}a_{2,1}a_{3,2} \\
& -b_{1,1}a_{2,3}a_{3,2} - b_{1,2}a_{2,1}a_{3,3} - b_{1,3}a_{2,2}a_{3,1}) + \\
& (c_{1,1}a_{2,2}a_{3,3} + c_{1,2}a_{2,3}a_{3,1} + c_{1,3}a_{2,1}a_{3,2} \\
& -c_{1,1}a_{2,3}a_{3,2} - c_{1,2}a_{2,1}a_{3,3} - c_{1,3}a_{2,2}a_{3,1}) \\
= \quad & \begin{vmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} + \begin{vmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}
\end{aligned}$$

$(*)$ … distributive, associative a commutative axioms were used for the algebraic manipulation with these terms.

Linearity of the determinant

Theorem: The determinant of a matrix is linearly dependent on each its row and column.

Corollary: Addition of a scalar multiple of a row to another does not change the determinant; analogously for columns.

Informal proof:

$$\begin{vmatrix} — & a_{i,\bullet} + t \cdot a_{j,\bullet} & — \\ — & a_{j,\bullet} & — \end{vmatrix} = \begin{vmatrix} — & a_{i,\bullet} & — \\ — & a_{j,\bullet} & — \end{vmatrix} + t \cdot \begin{vmatrix} — & a_{j,\bullet} & — \\ — & a_{j,\bullet} & — \end{vmatrix} = \begin{vmatrix} — & a_{i,\bullet} & — \\ — & a_{j,\bullet} & — \end{vmatrix}$$

Corollary: If $\boldsymbol{A}$ is singular then $\det(\boldsymbol{A}) = 0$.

Proof: The dependent row can be eliminated to the zero row.

Determinant calculation

Transformation into the row echelon form over $\mathbb{Z}_5$:

$$\begin{vmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & 3 & 0 \\ 4 & 1 & 3 & 1 \\ 0 & 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 & 2 \\ 0 & 4 & 2 & 3 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2$$

Transformations used:

1. addition of the 3-multiple of the first row to the second and addition (of the 1-multiple) of the first row to the third
2. rearrangements of the rows according to the permutation with cycles $((1), (2, 3, 4))$ — does not change the sign
3. adding the third *column* to the second

Determinant of products

Theorem: For any $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{K}^{n \times n}$: $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \det(\boldsymbol{B})$.

Proof: W.l.o.g. both $\boldsymbol{A}$ or $\boldsymbol{B}$ are regular otherwise we get $0 = 0$.

Products with elementary matrices preserve determinant $\det(\boldsymbol{EB}) = \det(\boldsymbol{E}) \det(\boldsymbol{B})$, because:

- for addition of the i-th row to the j-th: $\det(\boldsymbol{E}) = 1$,
- for scaling of the i-th row by t: $\det(\boldsymbol{E}) = t$.

(The other operations can be derived from these two.)

Factorize the regular $\boldsymbol{A}$ into elementary matrices $\boldsymbol{A} = \boldsymbol{E}_1 \ldots \boldsymbol{E}_k$.
$\det(\boldsymbol{AB}) = \det(\boldsymbol{E}_1 \ldots \boldsymbol{E}_k \boldsymbol{B}) = \det(\boldsymbol{E}_1) \det(\boldsymbol{E}_2 \ldots \boldsymbol{E}_k \boldsymbol{B}) =$
$\det(\boldsymbol{E}_1) \ldots \det(\boldsymbol{E}_k) \det(\boldsymbol{B}) = \det(\boldsymbol{E}_1 \ldots \boldsymbol{E}_k) \det(\boldsymbol{B}) = \det(\boldsymbol{A}) \det(\boldsymbol{B})$

Corollary: $\det(\boldsymbol{A}^{-1}) = (\det(\boldsymbol{A}))^{-1}$.

Proof: $\det(\boldsymbol{A}) \det(\boldsymbol{A}^{-1}) = \det(\boldsymbol{AA}^{-1}) = \det(\boldsymbol{I}_n) = 1$

Corollary: $\boldsymbol{A}$ is regular *if and only if* $\det(\boldsymbol{A}) \neq 0$.

Laplace expansion

Notation: $\boldsymbol{A}^{i,j}$ is the submatrix obtained from $\boldsymbol{A}$ by deleting the i-th row and j-th column.

Theorem: For any $\boldsymbol{A} \in \mathbb{K}^{n \times n}$ and any $i \in \{1, \ldots, n\}$ it holds that:

$$\det(\boldsymbol{A}) = \sum_{j=1}^{n} a_{i,j}(-1)^{i+j} \det(\boldsymbol{A}^{i,j})$$

Proof: Express the i-th row as the linear combination of vectors of the standard basis (transposed to rows) and use the linearity:

$(a_{i,1}, a_{i,2}, \ldots, a_{i,n}) = a_{i,1}(\boldsymbol{e}^1)^T + a_{i,2}(\boldsymbol{e}^2)^T + \cdots + a_{i,n}(\boldsymbol{e}^n)^T$

$$\begin{vmatrix} \cdots\cdots\cdots \\ a_{i,1} \; a_{i,2} \; \ldots \; a_{i,n} \\ \cdots\cdots\cdots \end{vmatrix} = a_{i,1} \begin{vmatrix} \cdots\cdots \\ 1 \; 0 \ldots 0 \\ \cdots\cdots \end{vmatrix} + a_{i,2} \begin{vmatrix} \cdots\cdots \\ 0 \; 1 \; 0 \ldots 0 \\ \cdots\cdots \end{vmatrix} + \cdots + a_{i,n} \begin{vmatrix} \cdots\cdots \\ 0 \ldots 0 \; 1 \\ \cdots\cdots \end{vmatrix}$$

The j-th term: $\begin{vmatrix} \cdots\cdots\cdots \\ 0 \ldots 0 \; 1 \; 0 \ldots 0 \\ \cdots\cdots\cdots \end{vmatrix} = \begin{vmatrix} \cdots\cdots \\ -(\boldsymbol{e}^j)^T- \\ \cdots\cdots \end{vmatrix} = (-1)^{i+1} \begin{vmatrix} -(\boldsymbol{e}^j)^T- \\ \cdots\cdots \\ \cdots\cdots \end{vmatrix} =$

$= (-1)^{i+1+j+1} \begin{vmatrix} -(\boldsymbol{e}^1)^T- \\ \cdots\cdots \\ \cdots\cdots \end{vmatrix} = (-1)^{i+j} \begin{vmatrix} \begin{array}{|c|c|} \hline 1 & \boldsymbol{0}^T \\ \hline \boldsymbol{0} & \boldsymbol{A}^{i,j} \\ \hline \end{array} \end{vmatrix} = (-1)^{i+j} \det(\boldsymbol{A}^{i,j})$

Example of the Laplace expansion

Expansion along the first row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 0 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= -3 - 2 \cdot (-6) + 3 \cdot (-3) = 0$$

To determine the sign of the second determinant:

$$\begin{vmatrix} 0 & 2 & 0 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = - \begin{vmatrix} 2 & 0 & 0 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{vmatrix} = - \begin{vmatrix} 2 & 0 & 0 \\ . & 4 & 6 \\ . & 7 & 9 \end{vmatrix} = -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$

1. column swap by the transposition $(1, 2)$ changes the sign
2. the rest of the first row does not affect the determinant
3. the fixed element (1) is excluded from all permutations and the matrix order is reduced by one

The adjoint matrix

Definition: For a matrix $\boldsymbol{A} \in \mathbb{K}^{n\times n}$ the *adjoint matrix* is $\operatorname{adj}(\boldsymbol{A})$ defined as $\operatorname{adj}(\boldsymbol{A})_{j,i} = (-1)^{i+j} \det(\boldsymbol{A}^{i,j})$.

... the factors of the Laplace expansion along the i-th *row* of $\boldsymbol{A}$ we put into the i-th *column* of $\operatorname{adj}(\boldsymbol{A})$.

$$|\boldsymbol{A}| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 3 & 5 & 3 \end{vmatrix} = 2\cdot(-1)^{2+1} \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} + 3\cdot(-1)^{2+2} \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix} + 0\cdot(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix}$$

$$\operatorname{adj}(\boldsymbol{A})_{1,2} = (-1)^{2+1} \begin{vmatrix} . & 2 & 5 \\ * & . & . \\ . & 5 & 3 \end{vmatrix} = - \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} = 19$$

$$\operatorname{adj}(\boldsymbol{A}) = \begin{pmatrix} 9 & 19 & -15 \\ -6 & -12 & 10 \\ 1 & 1 & -1 \end{pmatrix}$$

The adjoint matrix and the inverse matrix

Theorem: For any regular matrix $\boldsymbol{A} \in \mathbb{K}^{n \times n}$: $\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})} \operatorname{adj}(\boldsymbol{A})$.

Example:

$$|\boldsymbol{A}| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 3 & 5 & 3 \end{vmatrix} = 9 + 50 + 0 - 45 - 0 - 12 = 2$$

$$\operatorname{adj}(\boldsymbol{A}) = \begin{pmatrix} 9 & 19 & -15 \\ -6 & -12 & 10 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})} \operatorname{adj}(\boldsymbol{A}) = \begin{pmatrix} 9/2 & 19/2 & -15/2 \\ -3 & -6 & 5 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$$

The adjoint matrix and the inverse matrix

Theorem: For any regular matrix $\boldsymbol{A} \in \mathbb{K}^{n \times n} : \boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})} \operatorname{adj}(\boldsymbol{A})$.

Proof: By the Laplace expansion of $\det(\boldsymbol{A})$:

$$(i\text{-th row of } \boldsymbol{A}) \cdot (i\text{-th column of } \operatorname{adj}(\boldsymbol{A})) = \det(\boldsymbol{A})$$

for $j \neq i$: $(j\text{-th row of } \boldsymbol{A}) \cdot (i\text{-th column of } \operatorname{adj}(\boldsymbol{A})) = \det(\boldsymbol{A}') = 0$,
as $\boldsymbol{A}'$ is obtained from $\boldsymbol{A}$ by replacing the i-th row by the j-th.

Thus: $\boldsymbol{A} \cdot \operatorname{adj}(\boldsymbol{A}) = \det(\boldsymbol{A}) \cdot \boldsymbol{I}_n \Rightarrow \boldsymbol{A} \cdot \left(\frac{1}{\det(\boldsymbol{A})} \operatorname{adj}(\boldsymbol{A})\right) = \boldsymbol{I}_n$

Example: The entry on the diagonal for $i = 2$: $(\boldsymbol{A} \cdot \operatorname{adj}(\boldsymbol{A}))_{2,2} =$

$$2 \cdot (-1)^{2+1} \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} + 3 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix} + 0 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 3 & 5 & 3 \end{vmatrix} = |\boldsymbol{A}|$$

Off the diagonal for $i = 2$ and $j = 1$: $(\boldsymbol{A} \cdot \operatorname{adj}(\boldsymbol{A}))_{2,1} =$

$$1 \cdot (-1)^{2+1} \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} + 2 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix} + 5 \cdot (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 3 & 5 & 3 \end{vmatrix} = 0$$

Cramer rule

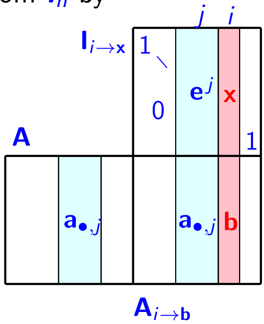
Theorem: Let $\boldsymbol{A} \in \mathbb{K}^{n \times n}$ be a regular matrix. For any $\boldsymbol{b} \in \mathbb{K}^n$, the solution $\boldsymbol{x}$ of $\boldsymbol{Ax} = \boldsymbol{b}$ satisfies $x_i = \frac{1}{\det(\boldsymbol{A})} \det(\boldsymbol{A}_{i \to \boldsymbol{b}})$, where $\boldsymbol{A}_{i \to \boldsymbol{b}}$ is the matrix obtained from $\boldsymbol{A}$ by replacing its i-th column with the vector $\boldsymbol{b}$.

Proof: Consider the matrix $\boldsymbol{I}_{i \to \boldsymbol{x}}$ obtained from $\boldsymbol{I}_n$ by replacing its i-th column with the vector $\boldsymbol{x}$.

Then $\boldsymbol{A} \cdot \boldsymbol{I}_{i \to \boldsymbol{x}} = \boldsymbol{A}_{i \to \boldsymbol{b}}$,

thus $\det(\boldsymbol{A}) \cdot \det(\boldsymbol{I}_{i \to \boldsymbol{x}}) = \det(\boldsymbol{A}_{i \to \boldsymbol{b}})$,

hence $x_i = \det(\boldsymbol{I}_{i \to \boldsymbol{x}}) = \frac{1}{\det(\boldsymbol{A})} \det(\boldsymbol{A}_{i \to \boldsymbol{b}})$.



Cramer rule — example

The system $\boldsymbol{Ax} = \boldsymbol{b} = (7,4,9)^T$ can be solved by determinants:

$$\det(\boldsymbol{A}_{1\to\boldsymbol{b}}) = \begin{vmatrix} 7 & 2 & 5 \\ 4 & 3 & 0 \\ 9 & 5 & 3 \end{vmatrix} = 4, \qquad \det(\boldsymbol{A}_{2\to\boldsymbol{b}}) = \begin{vmatrix} 1 & 7 & 5 \\ 2 & 4 & 0 \\ 3 & 9 & 3 \end{vmatrix} = 0,$$

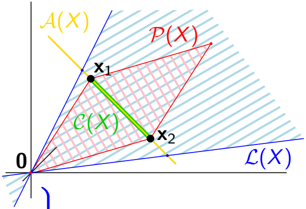
$$\det(\boldsymbol{A}_{3\to\boldsymbol{b}}) = \begin{vmatrix} 1 & 2 & 7 \\ 2 & 3 & 4 \\ 3 & 5 & 9 \end{vmatrix} = 2$$

Hence $\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})} \begin{pmatrix} \det(\boldsymbol{A}_{1\to\boldsymbol{b}}) \\ \det(\boldsymbol{A}_{2\to\boldsymbol{b}}) \\ \det(\boldsymbol{A}_{3\to\boldsymbol{b}}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

Check:

$$\boldsymbol{Ax} = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 & 0 \\ 3 & 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 9 \end{pmatrix} = \boldsymbol{b}$$

Different kinds of hull of a set in the Euclidean space



For a set $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$

Linear hull: $\mathcal{L}(X) = \left\{ \sum_{i=1}^{k} a_i \boldsymbol{x}_i, a_i \in \mathbb{R} \right\}$

. . . the smallest subspace containing X.

Affine hull: $\mathcal{A}(X) = \left\{ \sum_{i=1}^{k} a_i \boldsymbol{x}_i, a_i \in \mathbb{R}, \sum_{i=1}^{k} a_i = 1 \right\}$

. . . the smallest translation-of-a-subspace containing X.

Convex hull: $\mathcal{C}(X) = \left\{ \sum_{i=1}^{k} a_i \boldsymbol{x}_i, a_i \in [0,1], \sum_{i=1}^{k} a_i = 1 \right\}$

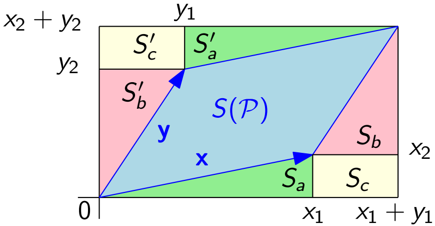
. . . the smallest convex set containing X.

The parallelepiped spanned by X: $\mathcal{P}(X) = \left\{ \sum_{i=1}^{k} a_i \boldsymbol{x}_i, a_i \in [0,1] \right\}$

Geometric meaning of the determinant

Theorem: Given vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^n$, then the volume of the parallelepiped $\mathcal{P}$ spanned by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is $|\det(\boldsymbol{A})|$, where the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ form the columns of $\boldsymbol{A}$.

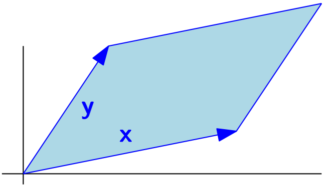
Example: The area $S(\mathcal{P})$ of a parallelogram spanned by two vectors $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^2$:



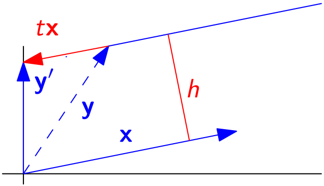
$$
\begin{aligned}
S(\mathcal{P}) &= (x_1 + y_1)(x_2 + y_2) - 2(S_a + S_b + S_c) \\
&= x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2 - x_1x_2 - y_1y_2 - 2y_1x_2 \\
&= x_1y_2 - y_1x_2 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}
\end{aligned}
$$

Proof idea - elementary transforms preserve the volume

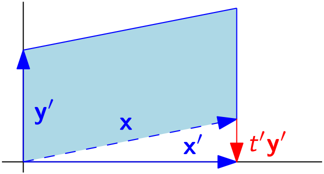
Applied on the transpose $\det(\boldsymbol{A}^T) = \det(\boldsymbol{A})$; vectors are *rows*.



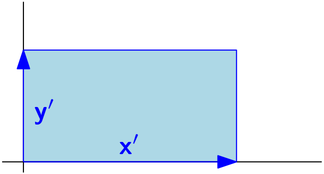
$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$



$$\begin{vmatrix} x_1 & x_2 \\ y_1 + tx_1 & y_2 + tx_2 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 \\ 0 & y_2' \end{vmatrix}$$



$$\begin{vmatrix} x_1 + t'0 & x_2 + t'y_2' \\ 0 & y_2' \end{vmatrix} = \begin{vmatrix} x_1' & 0 \\ 0 & y_2' \end{vmatrix}$$



$$\begin{vmatrix} x_1' & 0 \\ 0 & y_2' \end{vmatrix}$$

Geometric meaning of the determinant

Theorem: Given vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^n$, then the volume of the parallelepiped $\mathcal{P}$ spanned by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is $|\det(\boldsymbol{A})|$, where the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ form the columns of $\boldsymbol{A}$.

Corollary: For a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ and $[f]_{XX}$ is the matrix of this linear map w.r.t. some basis X, then the volumes of bodies change under f as follows:

$$vol(f(V)) = |\det([f]_{XX})| \cdot vol(V)$$

Proof idea: Split V into axis aligned hypercubes, then they are mapped onto parallelepipeds with volumes changed by the factor $|\det([f]_{KK})|$, because the matrix $[f]_{KK}$ contains images of vectors of the standard basis as its columns.

For other bases: $\det([f]_{XX}) = \det([id]_{KX}[f]_{KK}[id]_{XK}) =$
$\det([id]_{XK})^{-1} \det([f]_{KK}) \det([id]_{XK}) = \det([f]_{KK})$.