Name: .... Sample test ....

1. (a) Define the inverse matrix.
   Find matrices $\mathbf{A}, \mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}_n$ but $\mathbf{A} \neq \mathbf{B}^{-1}$.

   (b) Define a transposition.
   Determine the sign of a permutation $(5, 3, 4, 1, 7, 6, 2)$ with the help of transpositions.

   (c) Define the vector of coordinates.
   In the space of real polynomials of degree at most four determine the vector of coordinates $[f]_X$ of the vector $f(x) = 3x^4+3x^3+x+3$ with respect to the basis $X = (x^4+x^3,\ x^3+x^2,\ x^2+x,\ x+1,\ x^4+1)$.

2. State and prove a theorem about the relationship between solutions of $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ax} = \mathbf{0}$.

3. Write a summary about vector spaces and their subspaces.

   (Please provide definitions, theorem statements, examples and relationships. Proofs are not required.)

1 a) B is the inverse matrix of a square matrix A if $AB = I_n$.

Example.. A cannot be square .. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $AB = I_2$.

b) A transposition is a permutation with a single cycle of length 2 an $n-2$ cycles of length 1.

The problem: the permutation consists of a single nontrivial cycle $(1, 5, 7, 2, 3, 4)$ and a fixed point (6). The cycle decomposes as.

$(1,4) \circ (1,3) \circ (1,2) \circ (1,7) \circ (1,5)$.

$\operatorname{Sign}(\pi) = -1^{\#\text{transpositions}} = -1^{5} = -1$.

c) If X is a finite basis of V, $X = (v_1, v_2, \dots, v_n)$

then the vector of coordinates of u w.r.t. X is

$$[u]_X = (a_1, \dots, a_n)^T \in \mathbb{K}^n \quad \text{where} \quad u = \sum_{i=1}^{n} a_i v_i .$$

The problem: solve $3x^4 + 3x^3 + x + 3 = a_1(x^4+x^3) + a_2(x^3+x^2) + a_3(x^2+x) + a_4(x+1) + a_5(x^4+1)$

This leads to a system with matrix:

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{array}\right) \Rightarrow \begin{array}{l} a_1 = 2 \\ a_2 = 1 \\ a_3 = -1 \\ a_4 = 2 \\ a_5 = 1 \end{array} \Rightarrow [f]_X = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$$

2. If the system $Ax = b$ has at least one solution x_0

then the map $f: x \to x + x_0$ is a bijection between

the sets of solutions of $Ax = 0$ and $Ax = b$.

Proof: a) x is a solution of $Ax = 0 \Rightarrow x + x_0$ is a solution of $Ax = b$:

$$A f(x) = A(x + x_0) = Ax + Ax_0 = 0 + b = b.$$

b) $f(x)$ is a solution of $Ax = b \Rightarrow x$ is a solution of $Ax = 0$.

$$Ax = Ax + Ax_0 - b = A(x + x_0) - b = A f(x) - b = b - b = 0 .$$

c) $x \neq x' \Rightarrow f(x) = x + x_0 \neq x' + x_0 = f(x') \Rightarrow f$ is injective

d) $\forall x: Ax = b: \; f^{-1}(x) := x - x_0 \in \mathrm{Ker}(A)$ since

$$A f^{-1}(x) = A(x - x_0) = Ax - Ax_0 = b - b = 0 \Rightarrow f \text{ is surjective}$$

$\boxtimes$

3. Definition: A vector space over a field $\mathbb{K}$ is $(V, +, \cdot)$ where $(V, +)$ is an Abelian group, $\cdot : \mathbb{K} \times V \to V$ satisfying:

1) $\forall u \in V: \ 1 \cdot u = u$, where 1 is the neutral element of $(\mathbb{K}, \cdot)$
2) $\forall a, b \in \mathbb{K}, \forall u \in V: (a \cdot b) \cdot u = a \cdot (b \cdot u)$
3) $\forall a, b \in \mathbb{K}, \forall u \in V: (a + b) \cdot u = a \cdot u + b \cdot u$
4) $\forall a \in \mathbb{K}, \forall u, v \in V: a \cdot (u + v) = a \cdot u + a \cdot v$.

Examples: Arithmetic vector space $\mathbb{K}^n$

- ordered n-tuples of elements of $\mathbb{K}$, $+, \cdot$ coordinate-wise

Analogously (real) sequences / functions on the same domain

in particular polynomials, continuous functions.

All subsets of a set as a vector space over $\mathbb{Z}_2$

$+$ .. symmetric difference

The set of even subgraphs of a fixed graph

$+$ ... —"— , over $\mathbb{Z}_2$.

Definition U is a subspace of V if

1) $\forall u, v \in U: \ u + v \in U$
2) $\forall u \in U \ \forall a \in \mathbb{K}: \ a \cdot u \in U$.

Facts: A subspace is also a vector space over the same field.

The zero vector of V belongs to all its subspaces.

Theorem: Let $\{U_i, i \in I\}$ be a collection of subspaces of V. Then $\bigcap_{i \in I} U_i$ is a subspace of V.

Definition: For a set $X \subseteq V$ let $\mathcal{L}(X)$, the subspace generated by X, be the intersection of all subspaces containing X.

Definition: A vector u is a linear combination of vectors from X if $\exists k, x_1, \dots, x_k \in X, a_1, \dots, a_k \in \mathbb{K}$ s.t. $u = \sum_{i=1}^{k} a_i x_i$

Theorem: For any V, any $X \subseteq V$ it holds that $\mathcal{L}(X)$ consists of all linear combinations of X.

Formally: $\bigcap \{U : X \subseteq U\} = \{u : u = \sum_{i=1}^{k} a_i x_i, x_i \in X, a_i \in \mathbb{K}, k \in \mathbb{N}$

Examples for $V = \mathbb{R}^3$, $u \neq 0$, $\mathcal{L}(u)$ is the line passing through u and 0.

Analogously for $0, u, v$ non-colinear $\mathcal{L}(\{u, v\})$ is the plane containing u, v and 0.

For $A \in \mathbb{K}^{m \times n}$, $\operatorname{Ker}(A) = \{x : Ax = 0\} \subseteq \mathbb{K}^n$.

$\mathcal{R}(A) = \mathcal{L}(\text{rows of } A) \subseteq \mathbb{K}^n$

$\mathcal{C}(A) = \mathcal{L}(\text{columns of } A) \subseteq \mathbb{K}^m$