

## A geometric problem

**Problem:** Can a rectangle with irrational ratio of the length of its sides be partitioned into finitely many squares?

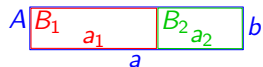
**Note:** For a rational ratio  $p : q$  we may use  $pq$  squares  $1 \times 1$ .

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**Theorem:** For an irrational ratio no such partition exists.

**Proof:** Let the rectangle  $R$  has side lengths  $1 : x$ , where  $x \in \mathbb{R} \setminus \mathbb{Q}$ . View  $\mathbb{R}$  as a vector space over  $\mathbb{Q}$ . Here  $1$  and  $x$  are lin. independent. Choose any linear map  $f : \mathbb{R} \rightarrow \mathbb{R}$ , where  $f(1) = 1$  and  $f(x) = -1$ . Such map  $f$  exists as the image of a part of a basis is prescribed. For any rectangle  $A$  of side lengths  $a, b$  define  $v(A) = f(a)f(b)$ . If we cut  $A$  into rectangles  $B_1$  and  $B_2$  then  $v(A) = v(B_1) + v(B_2)$ .  
 $f(a)f(b) = f(a_1 + a_2)f(b) = (f(a_1) + f(a_2))f(b) = f(a_1)f(b) + f(a_2)f(b)$



Thus if we iteratively cut  $A$  into  $B_1, \dots, B_k$  then  $v(A) = \sum_{i=1}^k v(B_i)$ .

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If by a contradiction  $R$  could be partitioned into squares  $A_1, \dots, A_k$  of side lengths  $a_1, \dots, a_k$ , then we refine the partition along their sides to rectangles  $B_1, \dots, B_l$ . Now:

$$-1 = f(1)f(x) = v(R) = \sum_{j=1}^l v(B_j) = \sum_{i=1}^k v(A_i) = \sum_{i=1}^k f(a_i)^2 \geq 0$$

