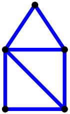
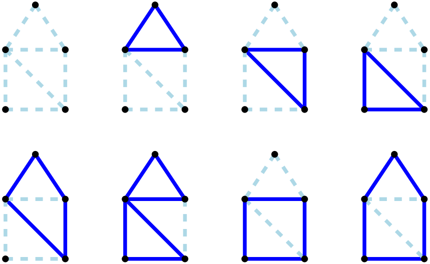
Even subgraphs

Let G be a connected graph and U contain edge sets A such that every vertex of G is belongs to an even number of edges in A. Such sets A yield the so called *even subgraphs* of G.

G



U

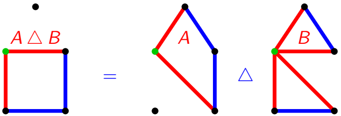


Problem: How many even subgraphs does G contain?

Vector space of even subgraphs

The symmetric difference $\triangle$ preserves even degrees, because the symmetric difference of two sets of even cardinality, namely the edges incident to a vertex, has also an even cardinality.

$$|A \triangle A| = |A| + |A| - 2|A \cap A|$$



Hence $(\mathcal{A}, \triangle, \cdot)$ forms a vector space over $\mathbb{Z}_2$.

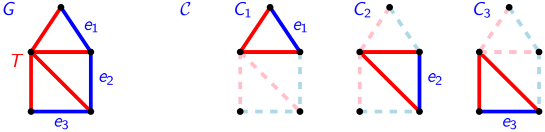
Vector spaces of finite cardinality satisfy $|\mathcal{A}| = |F|^{\dim(\mathcal{A})}$.

Equivalent problem: Construct a basis of $\mathcal{A}$.

Construction of a basis $\mathcal{C}$

For any spanning tree T of graph G, let $\{e_1, \ldots, e_k\} = E_G \setminus E_T$.
For each $i \in \{1, \ldots, k\}$ let C_i be the unique cycle in $T \cup e_i$.

The set $\mathcal{C} = \{C_1, \ldots, C_k\}$ is *linearly independent*, since for every i, the edge e_i is in C_i, but cannot be eliminated by the symmetric difference of C_i with other cycles from $\mathcal{C}$, as they do not have e_i.



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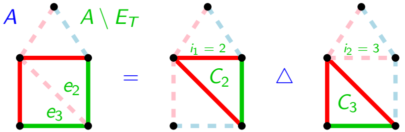
For any even subgraph A, let $\{e_{i_1}, \ldots, e_{i_l}\} = A \setminus E_T$, i.e., $i_1, \ldots, i_l \in \{1, \ldots, k\}$ are indices of edges from A but not in T.

The graph $A \triangle C_{i_1} \triangle C_{i_2} \triangle \cdots \triangle C_{i_l}$ is an even subgraph of G, but it is also a subgraph of T, since it has no edges from $E_G \setminus E_T$.
The tree has no cycles, so $A \triangle C_{i_1} \triangle C_{i_2} \triangle \cdots \triangle C_{i_l} = \mathbf{0}$.

From the equality follows
$A = C_{i_1} \triangle C_{i_2} \triangle \cdots \triangle C_{i_l}$.
Therefore, $\mathcal{C}$ *generates* $\mathcal{A}$.



Answer: Each connected graph G has $2^{|E_G|-|V_G|+1}$ even subgraphs.