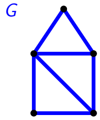
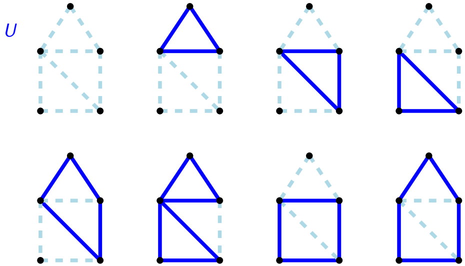
Even subgraphs

Let G be a connected graph and U contain edge sets A such that every vertex of G is belongs to an even number of edges in A. Such sets A yield the so called *even subgraphs* of G.



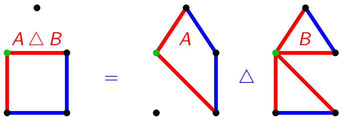


Problem: How many even subgraphs does G contain?

Vector space of even subgraphs

The symmetric difference $\triangle$ preserves even degrees, because the symmetric difference of two sets of even cardinality, namely the edges incident to a vertex, has also an even cardinality.

$$|A \triangle A| = |A| + |A| - 2|A \cap A|$$



Hence $(\mathcal{A}, \triangle, \cdot)$ forms a vector space over $\mathbb{Z}_2$.

Vector spaces of finite cardinality satisfy $|\mathcal{A}| = |F|^{\dim(\mathcal{A})}$.

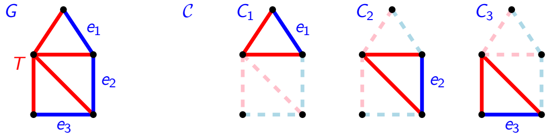
Equivalent problem: Construct a basis of $\mathcal{A}$.

Construction of a basis $\mathcal{C}$

Take an arbitrary spanning tree T of G.

For each $e_i \in E_G \setminus E_T$ define C_i as the unique cycle in $T \cup e$.

The set $\mathcal{C} = \{C_i : e_i \in E_G \setminus E_T\}$ is *linearly independent* since the edge e_i cannot be eliminated by the symmetric difference of C_i with other subgraphs in $\mathcal{C}$, because these do not contain e_i.



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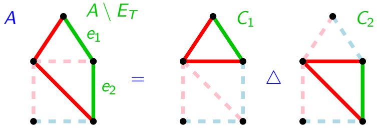
For an even subgraph A denote $A \setminus E_T = \{e_{i_1}, \dots, e_{i_k}\}$.
The graph $A \triangle C_{i_1} \triangle C_{i_2} \triangle \cdots \triangle C_{i_k}$ is an even subgraph of G, but also a subgraph of T, as it has no edge of $E_G \setminus E_T$.
A tree has no cycles, thus $A \triangle C_{i_1} \triangle C_{i_2} \triangle \cdots \triangle C_{i_k} = \mathbf{0}$.

Thus $A = C_{i_1} \triangle C_{i_2} \triangle \cdots \triangle C_{i_k}$. Hence the set $\mathcal{C}$ *generates* $\mathcal{A}$.



We get $\dim(\mathcal{A}) = |\mathcal{C}| = |E_G| - |E_T| = |E_G| - |V_G| + 1$.

Answer: Each connected graph G has $2^{|E_G|-|V_G|+1}$ even subgraphs.