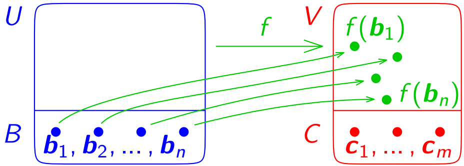
Matrix of a linear map

Definition: Let U and V be vector spaces over the same field F, with bases $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ and $C = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$.

The *matrix of a linear map* $f : U \to V$ w.r.t. bases B and C is $[f]_{B,C} \in F^{m \times n}$ whose columns are the vectors of coordinates with respect to the basis C of the images of the vectors of the basis B.

Formally: $$[f]_{B,C} = \begin{pmatrix} | & & | \\ [f(\boldsymbol{b}_1)]_C & \ldots & [f(\boldsymbol{b}_n)]_C \\ | & & | \end{pmatrix}.$$



$[f]_{B,C} = ([f(\boldsymbol{b}_1)]_C, \ldots, [f(\boldsymbol{b}_n)]_C)$

Use of the matrix of a linear map

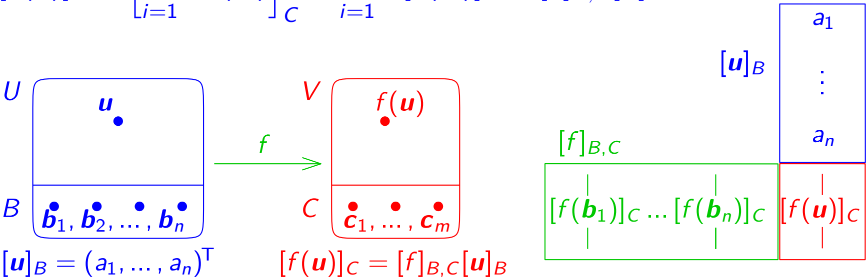
The matrix is $[f]_{B,C} = ([f(\boldsymbol{b}_1)]_C, \ldots, [f(\boldsymbol{b}_n)]_C)$.

Observation: For any $\boldsymbol{u} \in U$ it holds that: $[f(\boldsymbol{u})]_C = [f]_{B,C}[\boldsymbol{u}]_B$.

Proof: Let $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i$, i.e. $[\boldsymbol{u}]_B = (a_1, \ldots, a_n)^{\mathsf{T}}$.

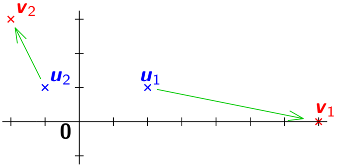
Then $f(\boldsymbol{u}) = f\left(\sum_{i=1}^{n} a_i \boldsymbol{b}_i\right) = \sum_{i=1}^{n} a_i f(\boldsymbol{b}_i)$ and hence also:

$$[f(\boldsymbol{u})]_C = \left[\sum_{i=1}^{n} a_i f(\boldsymbol{b}_i)\right]_C = \sum_{i=1}^{n} a_i [f(\boldsymbol{b}_i)]_C = [f]_{B,C}[\boldsymbol{u}]_B.$$



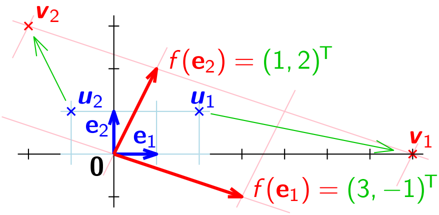
The matrix of a linear mapping in the plane

With respect to the standard basis E, find the matrix of $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\boldsymbol{u}_1 = (2,1)^\mathsf{T}$ on $\boldsymbol{v}_1 = (7,0)^\mathsf{T}$ and $\boldsymbol{u}_2 = (-1,1)^\mathsf{T}$ on $\boldsymbol{v}_2 = (-2,3)^\mathsf{T}$.



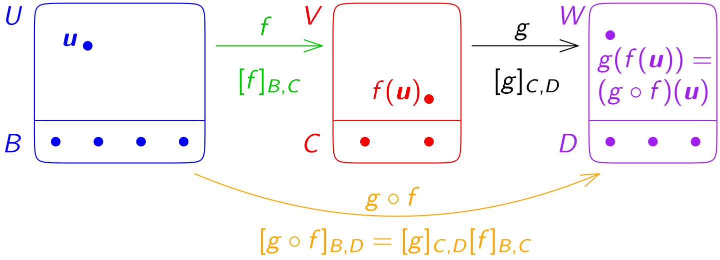
The matrix shall satisfy $[f]_{\mathsf{E},\mathsf{E}}[\boldsymbol{u}_i]_\mathsf{E} = [\boldsymbol{v}_i]_\mathsf{E}$ for $i \in \{1,2\}$, i.e.

$$[f]_{\mathsf{E},\mathsf{E}}\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}7 & -2\\ 0 & 3\end{pmatrix} \Rightarrow [f]_{\mathsf{E},\mathsf{E}} = \begin{pmatrix}7 & -2\\ 0 & 3\end{pmatrix}\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}.$$



Composition of linear maps

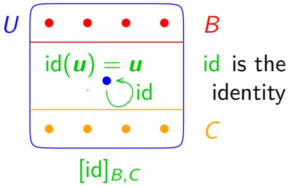
Observation: Let U, V and W be vector spaces over F with finite ordered bases B, C and D. For matrices of linear maps $f : U \to V$ and $g : V \to W$ it holds that: $[g \circ f]_{B,D} = [g]_{C,D}[f]_{B,C}$



Proof: For any $\boldsymbol{u} \in U : [(g \circ f)(\boldsymbol{u})]_D = [g \circ f]_{B,D}[\boldsymbol{u}]_B$, and also:
$[(g \circ f)(\boldsymbol{u})]_D = [g(f(\boldsymbol{u}))]_D = [g]_{C,D}[f(\boldsymbol{u})]_C = [g]_{C,D}[f]_{B,C}[\boldsymbol{u}]_B$.
If we substitute for $\boldsymbol{u}$ the i-th vector of B, we get $[\boldsymbol{u}]_B = \mathbf{e}_i$ and then $[g \circ f]_{B,D}\mathbf{e}_i = ([g]_{C,D}[f]_{B,C})\mathbf{e}_i$ yields that the matrices have the i-th columns identical. Therefeore $[g \circ f]_{B,D} = [g]_{C,D}[f]_{B,C}$.

The change of basis matrix

Definition: Let B and C be two finite ordered bases of a vector space U. The matrix $[\text{id}]_{B,C}$ is the *change of basis matrix* from B to C.



Observation: For every $\boldsymbol{u} \in U$ it holds:
$[\boldsymbol{u}]_C = [\text{id}(\boldsymbol{u})]_C = [\text{id}]_{B,C}[\boldsymbol{u}]_B$.

Observation: Since $[\text{id}]_{C,B}[\text{id}]_{B,C} = [\text{id}]_{B,B} = \mathbf{I}$,
every change of basis matrix is regular and $[\text{id}]_{C,B} = ([\text{id}]_{B,C})^{-1}$

Procedure: Calculation of $[\text{id}]_{B,C}$ from a basis B to a basis C in F^n:

For $B = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$ and $C = (\boldsymbol{c}_1, \dots, \boldsymbol{c}_n)$ build $\boldsymbol{B} = \begin{pmatrix} | & & | \\ \boldsymbol{b}_1 & \dots & \boldsymbol{b}_n \\ | & & | \end{pmatrix}$, $\boldsymbol{C} = \begin{pmatrix} | & & | \\ \boldsymbol{c}_1 & \dots & \boldsymbol{c}_n \\ | & & | \end{pmatrix}$.

Each $\boldsymbol{u} \in F^n$ has $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i = \boldsymbol{B}[\boldsymbol{u}]_B$ with $[\boldsymbol{u}]_B = (a_1, \dots, a_n)^\mathsf{T}$,

and also $\boldsymbol{u} = \sum_{i=1}^{n} d_i \boldsymbol{c}_i = \boldsymbol{C}[\boldsymbol{u}]_C$ for $[\boldsymbol{u}]_C = (d_1, \dots, d_n)^\mathsf{T}$.

Now $\boldsymbol{u} = \boldsymbol{B}[\boldsymbol{u}]_B = \boldsymbol{C}[\boldsymbol{u}]_C$ gives: $[\boldsymbol{u}]_C = \boldsymbol{C}^{-1}\boldsymbol{B}[\boldsymbol{u}]_B = [\text{id}]_{B,C}[\boldsymbol{u}]_B$.

Trick: Save the product by: $(\boldsymbol{C}|\boldsymbol{B}) \sim\sim (\mathbf{I}|\boldsymbol{C}^{-1}\boldsymbol{B}) = (\mathbf{I}|[\text{id}]_{B,C})$.

Example

In the space $\mathbb{Z}_5^4$ determine the change of basis matrix
from $B = \{(2,3,0,2)^\mathsf{T}, (1,1,1,1)^\mathsf{T}, (2,0,3,3)^\mathsf{T}, (1,4,2,0)^\mathsf{T}\}$
to $C = \{(1,2,0,1)^\mathsf{T}, (2,0,3,3)^\mathsf{T}, (3,1,4,1)^\mathsf{T}, (4,2,0,1)^\mathsf{T}\}$.

Form a matrix, the columns on the left side are from C,
on the right from B. By Gauss-Jordan elimination transform the left side into $\mathbf{I}$. The change of basis matrix $[\mathrm{id}]_{B,C}$ is on the right.

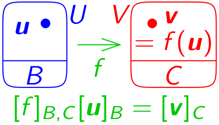
$$\left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 2 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 & 3 & 1 & 0 & 4 \\ 0 & 3 & 4 & 0 & 0 & 1 & 3 & 2 \\ 1 & 3 & 1 & 1 & 2 & 1 & 3 & 0 \end{array}\right) \sim\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 & 3 & 1 & 3 \\ 0 & 0 & 1 & 0 & 3 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 \end{array}\right)$$

The change of basis matrix from B to C is: $[\mathrm{id}]_{B,C} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 3 & 1 & 3 \\ 3 & 3 & 0 & 4 \\ 0 & 4 & 0 & 0 \end{pmatrix}$

Characterization of maps using matrices

Lemma: Linear map $f : U \to V$ between spaces U and V with arbitrary finite bases B and C

- is injective iff $\mathrm{rank}([f]_{B,C}) = \dim U$.
- is surjective iff $\mathrm{rank}([f]_{B,C}) = \dim V$,



Proof: Vectors $\boldsymbol{u}$ such that $f(\boldsymbol{u}) = \boldsymbol{v}$ correspond uniquely to solutions $\boldsymbol{x}$ of the system $[f]_{B,C}\boldsymbol{x} = [\boldsymbol{v}]_C$, by choosing $\boldsymbol{x} = [\boldsymbol{u}]_B$.

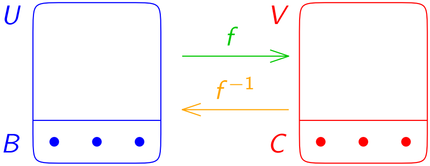
- The system has at most one solution for each $\boldsymbol{v} \in V$ iff the system $[f]_{B,C}\boldsymbol{x} = \mathbf{0}$ has no free variables, namely when $\mathrm{rank}([f]_{B,C})$ is equal to the number of columns, i.e., $\dim U$.
- It has a solution for every $\boldsymbol{v} \in V$ iff $\mathrm{rank}([f]_{B,C})$ is equal to the number of rows, i.e., $\dim V$. (No pivot on the right.)

Theorem: A linear map $f\colon U \to V$ is an isomorphism of spaces U and V with finite bases B and C, if and only if $[f]_{B,C}$ is regular.

Corollary: For an isomorphism f, we have: $[f^{-1}]_{C,B} = ([f]_{B,C})^{-1}$, as $[f]_{B,C}$ is square and satisfies $[f^{-1}]_{C,B}[f]_{B,C} = [\mathrm{id}]_{B,B} = \mathbf{I}$.

Bonus: A brief proof without using systems of equations

Theorem: A linear map $f : U \to V$ is an isomorphism of spaces U and V with finite bases B and C, if and only if $[f]_{B,C}$ is regular.

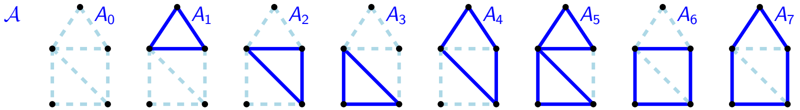


Proof: $\Leftarrow$: Choose $g : V \to U$ such that $[g]_{C,B} = ([f]_{B,C})^{-1}$. Then:
$[g \circ f]_{B,B} = [f]_{B,C}^{-1}[f]_{B,C} = \mathbf{I}_{|B|} = [\mathrm{id}]_{B,B} \Rightarrow f$ is injective,
$[f \circ g]_{C,C} = [f]_{B,C}[f]_{B,C}^{-1} = \mathbf{I}_{|C|} = [\mathrm{id}]_{C,C} \Rightarrow f$ is surjective.

$\Rightarrow$: Since $f(U) = V$ and $f^{-1}(V) = U$, we have $\dim(U) = \dim(V)$.
The matrix $[f]_{B,C}$ is square satisfying $[f^{-1}]_{C,B}[f]_{B,C} = [\mathrm{id}]_{B,B} = \mathbf{I}$.

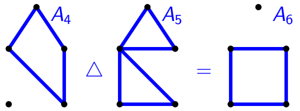
Example of an isomorphism

Let $(\mathcal{A}, \triangle)$ be the vector space of even subgraphs of a graph G. The underlying field is $\mathbb{Z}_2$. The map $f : \mathcal{A} \to \mathbb{Z}_2^3$ given in the table below is linear and bijective, hence an isomorphism.



$f(A_i)$: $(0,0,0)^{\mathsf{T}}$ $(1,0,0)^{\mathsf{T}}$ $(0,1,0)^{\mathsf{T}}$ $(0,0,1)^{\mathsf{T}}$ $(1,1,0)^{\mathsf{T}}$ $(1,0,1)^{\mathsf{T}}$ $(0,1,1)^{\mathsf{T}}$ $(1,1,1)^{\mathsf{T}}$

Linearity holds, e.g:



$(1,1,0)^{\mathsf{T}} + (1,0,1)^{\mathsf{T}} = (0,1,1)^{\mathsf{T}}$

$f(A_4) + f(A_5) = f(A_4 \triangle A_5) = f(A_6)$

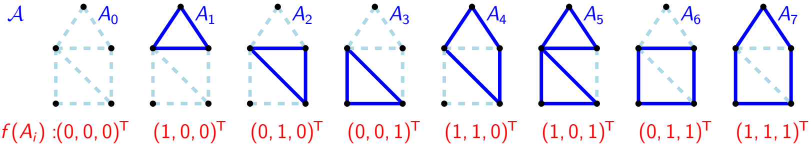
The matrix of the mapping depends on both bases chosen.

E.g. $[f]_{\{A_1, A_2, A_3\}, \mathsf{E}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The dimension of both spaces is 3.

Use of the matrix

For another choice $B = \{A_4, A_5, A_1\}$ we get $[f]_{B,\mathsf{E}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$



Observe that $[f]_{B,\mathsf{E}}[A]_B = [f(A)]_{\mathsf{E}}$ holds.
E.g. for A_6 we get: $A_6 = A_4 \triangle A_5$ and hence $[A_6]_B = (1, 1, 0)^{\mathsf{T}}$.
Now:

$$[f]_{B,\mathsf{E}}[A_6]_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = [f(A_6)]_{\mathsf{E}}$$

Questions to understand the lecture topic

- What can be said about mappings whose matrix is the identity matrix or a permutation matrix?
- Is it easier to determine the change of basis matrix from the standard basis or to the standard basis?
- What do we get if we multiply the matrix of the mapping $[f]_{B,C}$ by the matrix of the inverse mapping $[f^{-1}]_{D,B}$?
- How are the rank of the matrix of the mapping and the properties of the mapping related, if it is injective or surjective?
- If two isomorphisms between finite-dimensional spaces can be composed, will the resulting mapping again be an isomorphism?