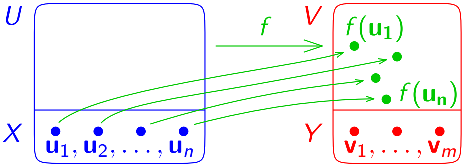
Matrix of a linear map

Definition: Let U and V be vector spaces over the same field $\mathbb{K}$, with bases $X = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$ and $Y = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$.

The *matrix of a linear map* $f : U \to V$ w.r.t. bases X and Y is $[f]_{X,Y} \in \mathbb{K}^{m\times n}$ whose columns are the vectors of coordinates with respect to the basis Y of the images of the vectors of the basis X.

Formally: $[f]_{X,Y} = \begin{pmatrix} | & & | \\ [f(\boldsymbol{u}_1)]_Y & \ldots & [f(\boldsymbol{u}_n)]_Y \\ | & & | \end{pmatrix}.$



$[f]_{X,Y} = ([f(\mathbf{u}_1)]_Y, \ldots, [f(\mathbf{u}_n)]_Y)$

Use of the matrix of a linear map

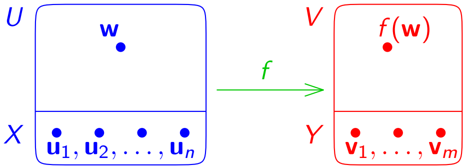
The matrix is $[f]_{X,Y} = ([f(\boldsymbol{u}_1)]_Y, \ldots, [f(\boldsymbol{u}_n)]_Y)$.

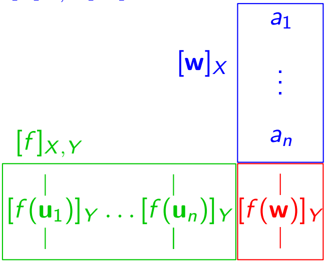
Observation: For any $\boldsymbol{w} \in U : [f(\boldsymbol{w})]_Y = [f]_{X,Y}[\boldsymbol{w}]_X$.

Proof: $\boldsymbol{w} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i$, i.e. $[\boldsymbol{w}]_X = (a_1, \ldots, a_n)^T$.

$$f(\boldsymbol{w}) = f\left(\sum_{i=1}^{n} a_i \boldsymbol{u}_i\right) = \sum_{i=1}^{n} a_i f(\boldsymbol{u}_i).$$

$$[f(\boldsymbol{w})]_Y = \left[\sum_{i=1}^{n} a_i f(\boldsymbol{u}_i)\right]_Y = \sum_{i=1}^{n} a_i [f(\boldsymbol{u}_i)]_Y = [f]_{X,Y}[\boldsymbol{w}]_X.$$

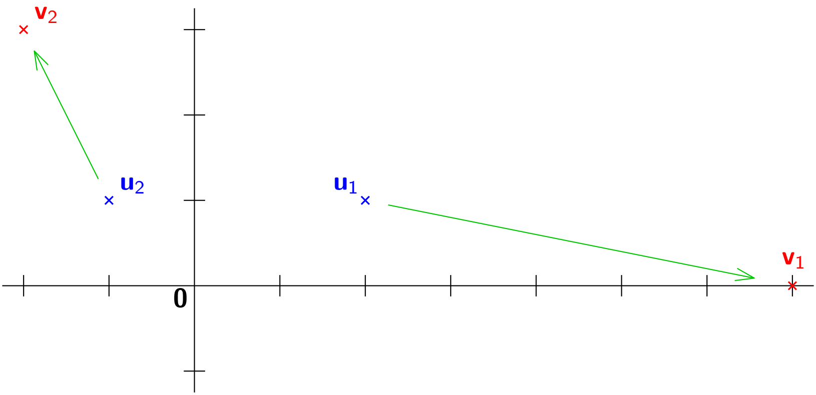




$[\mathbf{w}]_X = (a_1, \ldots, a_n)^T$ $\quad$ $[f(\mathbf{w})]_Y = [f]_{X,Y}[\mathbf{w}]_X$

The matrix of a linear mapping in the plane

Determine the matrix of a mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$
with respect to the standard basis K,
that maps vector $\boldsymbol{u}_1 = (2, 1)^T$ onto $\boldsymbol{v}_1 = (7, 0)^T$,
and also $\boldsymbol{u}_2 = (-1, 1)^T$ onto $\boldsymbol{v}_2 = (-2, 3)^T$.

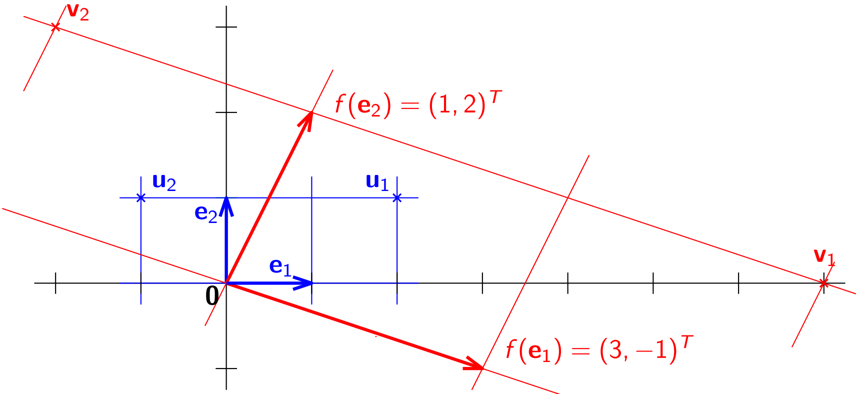


The matrix shall satisfy $[f]_{K,K}[\boldsymbol{u}_i]_K = [\boldsymbol{v}_i]_K$ for $i = 1, 2$, i.e.

$$[f]_{K,K}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 0 & 3 \end{pmatrix}$$

hence

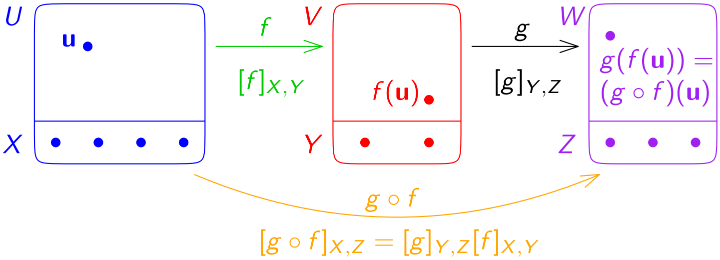
$$[f]_{K,K} = \begin{pmatrix} 7 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$$



Composition of linear maps

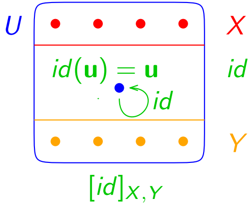
Observation: Let U, V and W be vector spaces over $\mathbb{K}$ with finite bases X, Y and Z. For matrices of linear maps $f : U \to V$ and $g : V \to W$ it holds that: $[g \circ f]_{X,Z} = [g]_{Y,Z}[f]_{X,Y}$

Proof: For any $\boldsymbol{u} \in U : [(g \circ f)(\boldsymbol{u})]_Z = [g \circ f]_{X,Z}[\boldsymbol{u}]_X$, also:
$[(g \circ f)(\boldsymbol{u})]_Z = [g(f(\boldsymbol{u}))]_Z = [g]_{Y,Z}[f(\boldsymbol{u})]_Y = [g]_{Y,Z}[f]_{X,Y}[\boldsymbol{u}]_X$,
hence $[g \circ f]_{X,Z} = [g]_{Y,Z}[f]_{X,Y}$.



The change of basis matrix

Definition: Let X and Y be two finite bases of a vector space U. The *change of basis matrix* from X to Y is $[id]_{X,Y}$.



id is the identity map

Observation: $[\boldsymbol{u}]_Y = [id(\boldsymbol{u})]_Y = [id]_{X,Y}[\boldsymbol{u}]_X$.

Observation: Since $[id]_{Y,X}[id]_{X,Y} = [id]_{X,X} = \boldsymbol{I}_n$, for $n = |X|$, every change of basis matrix is regular and $[id]_{Y,X} = ([id]_{X,Y})^{-1}$

Calculation of the change of basis matrix in $\mathbb{K}^n$

Let $X = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$ and $Y = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ be two bases of $\mathbb{K}^n$.

Build matrices $\boldsymbol{A} = \begin{pmatrix} | & & | \\ \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_n \\ | & & | \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} | & & | \\ \boldsymbol{v}_1 & \ldots & \boldsymbol{v}_n \\ | & & | \end{pmatrix}$

Any $\boldsymbol{w} \in \mathbb{K}^n : \boldsymbol{w} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i = \boldsymbol{A}[\boldsymbol{w}]_X$ with $[\boldsymbol{w}]_X = (a_1, \ldots, a_n)^T$.

similarly: $\boldsymbol{w} = \sum_{i=1}^{n} b_i \boldsymbol{v}_i = \boldsymbol{B}[\boldsymbol{w}]_Y$, with $[\boldsymbol{w}]_Y = (b_1, \ldots, b_n)^T$.

From $\boldsymbol{w} = \boldsymbol{A}[\boldsymbol{w}]_X = \boldsymbol{B}[\boldsymbol{w}]_Y$
we derive that: $[\boldsymbol{w}]_Y = \boldsymbol{B}^{-1}\boldsymbol{A}[\boldsymbol{w}]_X = [id]_{X,Y}[\boldsymbol{w}]_X$.

In practice we may use Gaussian elimination as follows:
$(\boldsymbol{B}|\boldsymbol{A}) \sim\sim (\boldsymbol{I}_n|\boldsymbol{B}^{-1}\boldsymbol{A}) = (\boldsymbol{I}_n|[id]_{X,Y})$.

Example

In the space $\mathbb{Z}_5^4$ determine the change of basis matrix
from $X = \{(2,3,0,2)^T, (1,1,1,1)^T, (2,0,3,3)^T, (1,4,2,0)^T\}$
to $Y = \{(1,2,0,1)^T, (2,0,3,3)^T, (3,1,4,1)^T, (4,2,0,1)^T\}$.

We form a matrix, the columns on the left side are from Y, on the right from X. By Gaussian elimination we transform the left side into $\boldsymbol{I}_n$. The change of basis matrix appears then on the right.

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 2 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 & 3 & 1 & 0 & 4 \\ 0 & 3 & 4 & 0 & 0 & 1 & 3 & 2 \\ 1 & 3 & 1 & 1 & 2 & 1 & 3 & 0 \end{array}\right) \sim\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 & 3 & 1 & 3 \\ 0 & 0 & 1 & 0 & 3 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 \end{array}\right)$$

The change of basis matrix from X to Y is: $[id]_{X,Y} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 3 & 1 & 3 \\ 3 & 3 & 0 & 4 \\ 0 & 4 & 0 & 0 \end{pmatrix}$

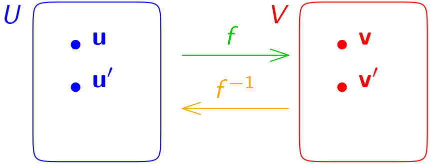
Isomorphism

Definition: A bijective linear map is called *isomorphism*.

Observation: If $f : U \to V$ is an isomorphism,
then $f^{-1} : V \to U$ is also an isomorphism.

Proof: f^{-1} is bijective by the definition, we show that it is linear.

For any $\boldsymbol{v}, \boldsymbol{v}' \in V$ let $\boldsymbol{u} = f^{-1}(\boldsymbol{v})$ and $\boldsymbol{u}' = f^{-1}(\boldsymbol{v}')$
i.e. $f(\boldsymbol{u}) = \boldsymbol{v}$ and $f(\boldsymbol{u}') = \boldsymbol{v}'$.

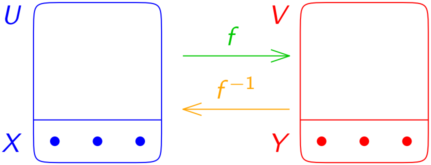


Linearity of addition: $f^{-1}(\boldsymbol{v} + \boldsymbol{v}') = f^{-1}(f(\boldsymbol{u}) + f(\boldsymbol{u}')) =$
$f^{-1}(f(\boldsymbol{u} + \boldsymbol{u}')) = \boldsymbol{u} + \boldsymbol{u}' = f^{-1}(\boldsymbol{v}) + f^{-1}(\boldsymbol{v}')$

Linearity of scalar multiples:
$\forall a \in \mathbb{K} : f^{-1}(a\boldsymbol{v}) = f^{-1}(af(\boldsymbol{u})) = f^{-1}(f(a\boldsymbol{u})) = a\boldsymbol{u} = af^{-1}(\boldsymbol{v})$.

Characterization of matrices of an isomorphism

Theorem: A linear map $f: U \to V$ is an isomorphism of spaces U and V with finite bases X and Y if and only if $[f]_{X,Y}$ is regular.



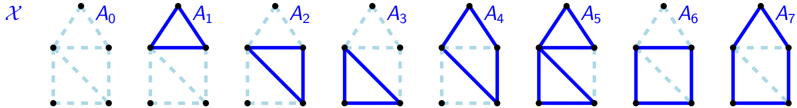
Proof: $\Leftarrow$: Consider $g: V \to U$ such that $[g]_{Y,X} = [f]^{-1}_{X,Y}$.
$[g \circ f]_{X,X} = [f]^{-1}_{X,Y}[f]_{X,Y} = \boldsymbol{I}_{|X|} = [id]_{X,X} \Rightarrow f$ is injective
$[f \circ g]_{Y,Y} = [f]_{X,Y}[f]^{-1}_{X,Y} = \boldsymbol{I}_{|Y|} = [id]_{Y,Y} \Rightarrow f$ is surjective

$$\Rightarrow: \left.\begin{array}{l} [f^{-1}]_{Y,X}[f]_{X,Y} = [id]_{X,X} = \boldsymbol{I}_{|X|} \Rightarrow |Y| \geq |X| \\ [f]_{X,Y}[f^{-1}]_{Y,X} = [id]_{Y,Y} = \boldsymbol{I}_{|Y|} \Rightarrow |X| \geq |Y| \end{array}\right\} \Rightarrow |X| = |Y|$$

Corollary: When f is an isomorphism: $[f^{-1}]_{Y,X} = [f]^{-1}_{X,Y}$.

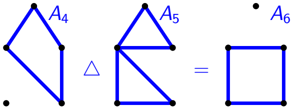
Example of an isomorphism

Let $(\mathcal{X}, \triangle)$ be the vector space of even subgraphs of a graph G. The underlying field is $\mathbb{Z}_2$. The following map $f : \mathcal{X} \to \mathbb{Z}_2^3$ is linear and bijective, hence an isomorphism.



$f(A_i) :(0,0,0)^T \quad (1,0,0)^T \quad (0,1,0)^T \quad (0,0,1)^T \quad (1,1,0)^T \quad (1,0,1)^T \quad (0,1,1)^T \quad (1,1,1)^T$

Linearity holds, e.g:



$(1,1,0)^T + (1,0,1)^T = (0,1,1)^T$

$f(A_4) + f(A_5) = f(A_4 \triangle A_5) = f(A_6)$

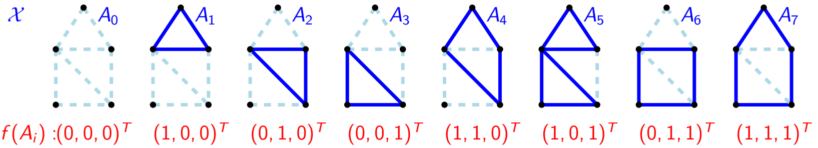
The matrix of the mapping depends on both bases chosen.

E.g. $[f]_{\{A_1,A_2,A_3\},K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The dimension of both spaces is 3.

Use of the matrix

For another choice $X = \{A_4, A_5, A_1\}$ we get $[f]_{X,K} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$



Observe that $[f]_{X,K}[A]_X = [f(A)]_K$ holds.
E.g. for A_6 we get: $A_6 = A_4 \triangle A_5$ and hence $[A_6]_X = (1,1,0)^T$.
Now:

$$[f]_{X,K}[A_6]_X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = [f(A_6)]_K$$