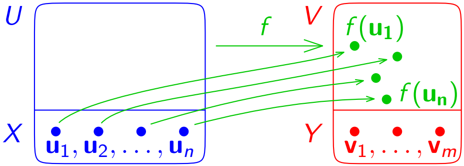
Matrix of a linear map

Definition: Let U and V be vector spaces over the same field $\mathbb{K}$, with bases $X = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$ and $Y = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$.

The *matrix of a linear map* $f : U \to V$ w.r.t. bases X and Y is $[f]_{X,Y} \in \mathbb{K}^{m \times n}$ whose columns are the vectors of coordinates with respect to the basis Y of the images of the vectors of the basis X.

Formally: $[f]_{X,Y} = \begin{pmatrix} | & & | \\ [f(\boldsymbol{u}_1)]_Y & \ldots & [f(\boldsymbol{u}_n)]_Y \\ | & & | \end{pmatrix}.$



$[f]_{X,Y} = ([f(\mathbf{u}_1)]_Y, \ldots, [f(\mathbf{u}_n)]_Y)$

Use of the matrix of a linear map

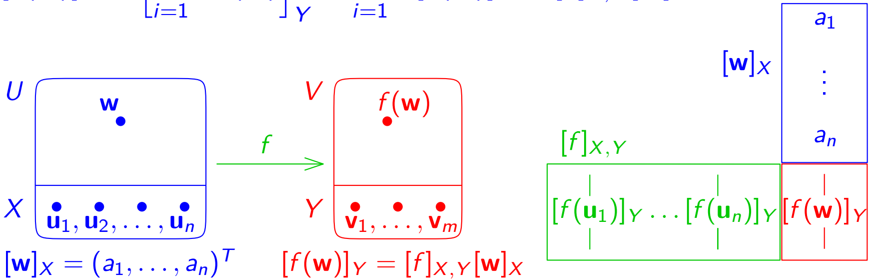
The matrix is $[f]_{X,Y} = ([f(\boldsymbol{u}_1)]_Y, \dots, [f(\boldsymbol{u}_n)]_Y)$.

Observation: For any $\boldsymbol{w} \in U$ it holds that: $[f(\boldsymbol{w})]_Y = [f]_{X,Y}[\boldsymbol{w}]_X$.

Proof: Let $\boldsymbol{w} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i$, i.e. $[\boldsymbol{w}]_X = (a_1, \dots, a_n)^T$.

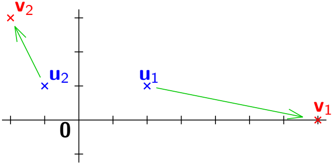
Then $f(\boldsymbol{w}) = f\left(\sum_{i=1}^{n} a_i \boldsymbol{u}_i\right) = \sum_{i=1}^{n} a_i f(\boldsymbol{u}_i)$ and hence also:

$[f(\boldsymbol{w})]_Y = \left[\sum_{i=1}^{n} a_i f(\boldsymbol{u}_i)\right]_Y = \sum_{i=1}^{n} a_i [f(\boldsymbol{u}_i)]_Y = [f]_{X,Y}[\boldsymbol{w}]_X$.



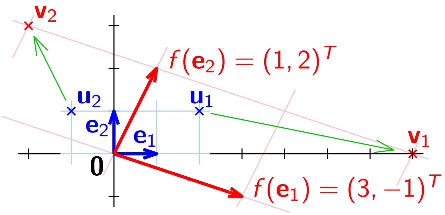
The matrix of a linear mapping in the plane

With respect to the standard basis K, find the matrix of $f : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\boldsymbol{u}_1 = (2,1)^T$ on $\boldsymbol{v}_1 = (7,0)^T$ and $\boldsymbol{u}_2 = (-1,1)^T$ on $\boldsymbol{v}_2 = (-2,3)^T$.



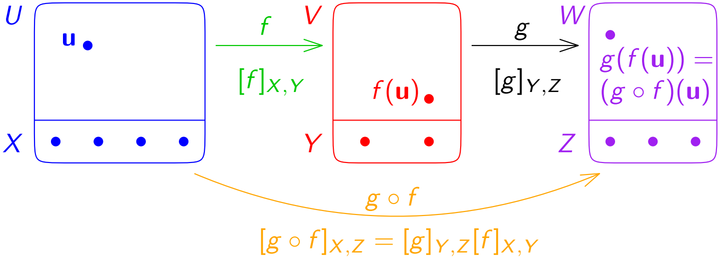
The matrix shall satisfy $[f]_{K,K}[\boldsymbol{u}_i]_K = [\boldsymbol{v}_i]_K$ for $i = 1, 2$, i.e.

$$[f]_{K,K}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -2 \\ 0 & 3 \end{pmatrix} \Rightarrow [f]_{K,K} = \begin{pmatrix} 7 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}.$$



Composition of linear maps

Observation: Let U, V and W be vector spaces over $\mathbb{K}$ with finite bases X, Y and Z. For matrices of linear maps $f : U \to V$ and $g : V \to W$ it holds that: $[g \circ f]_{X,Z} = [g]_{Y,Z}[f]_{X,Y}$



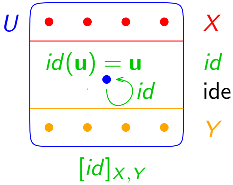
Proof: For any $\boldsymbol{u} \in U : [(g \circ f)(\boldsymbol{u})]_Z = [g \circ f]_{X,Z}[\boldsymbol{u}]_X$, and also:

$[(g \circ f)(\boldsymbol{u})]_Z = [g(f(\boldsymbol{u}))]_Z = [g]_{Y,Z}[f(\boldsymbol{u})]_Y = [g]_{Y,Z}[f]_{X,Y}[\boldsymbol{u}]_X$.

If we substitute for $\boldsymbol{u}$ the i-th vector of X, we get $[\boldsymbol{u}]_X = \boldsymbol{e}^i$ and then $[g \circ f]_{X,Z}\boldsymbol{e}^i = ([g]_{Y,Z}[f]_{X,Y})\boldsymbol{e}^i$ yields that the matrices have the i-th columns identical. Therefeore $[g \circ f]_{X,Z} = [g]_{Y,Z}[f]_{X,Y}$.

The change of basis matrix

Definition: Let X and Y be two finite bases of a vector space U. The matrix $[id]_{X,Y}$ is called the *change of basis matrix* from X to Y.



id is the identity

Observation: For every $\boldsymbol{u} \in U$ it holds:
$[\boldsymbol{u}]_Y = [id(\boldsymbol{u})]_Y = [id]_{X,Y}[\boldsymbol{u}]_X$.

Observation: Since $[id]_{Y,X}[id]_{X,Y} = [id]_{X,X} = \boldsymbol{I}_{\dim(U)}$,
every change of basis matrix is regular and $[id]_{Y,X} = ([id]_{X,Y})^{-1}$

Procedure: Calculation of $[id]_{X,Y}$ in $\mathbb{K}^n$:

For $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ and $Y = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)$ build $\boldsymbol{X} = \begin{pmatrix} | & & | \\ \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_n \\ | & & | \end{pmatrix}$, $\boldsymbol{Y} = \begin{pmatrix} | & & | \\ \boldsymbol{y}_1 & \ldots & \boldsymbol{y}_n \\ | & & | \end{pmatrix}$.

Each $\boldsymbol{u} \in \mathbb{K}^n$ has $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{x}_i = \boldsymbol{X}[\boldsymbol{u}]_X$ with $[\boldsymbol{u}]_X = (a_1, \ldots, a_n)^T$,

and also $\boldsymbol{u} = \sum_{i=1}^{n} b_i \boldsymbol{y}_i = \boldsymbol{Y}[\boldsymbol{u}]_Y$ for $[\boldsymbol{u}]_Y = (b_1, \ldots, b_n)^T$.

Now $\boldsymbol{u} = \boldsymbol{X}[\boldsymbol{u}]_X = \boldsymbol{Y}[\boldsymbol{u}]_Y$ gives: $[\boldsymbol{u}]_Y = \boldsymbol{Y}^{-1}\boldsymbol{X}[\boldsymbol{u}]_X = [id]_{X,Y}[\boldsymbol{u}]_X$.

Trick: Save the product by: $(\boldsymbol{Y}|\boldsymbol{X}) \sim\sim (\boldsymbol{I}_n|\boldsymbol{Y}^{-1}\boldsymbol{X}) = (\boldsymbol{I}_n|[id]_{X,Y})$.

Example

In the space $\mathbb{Z}_5^4$ determine the change of basis matrix
from $X = \{(2,3,0,2)^T, (1,1,1,1)^T, (2,0,3,3)^T, (1,4,2,0)^T\}$
to $Y = \{(1,2,0,1)^T, (2,0,3,3)^T, (3,1,4,1)^T, (4,2,0,1)^T\}$.

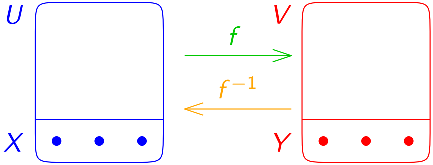
Form a matrix, the columns on the left side are from Y,
on the right from X. By Gauss-Jordan elimination transform the left side into $\boldsymbol{I_n}$. The change of basis matrix $[id]_{X,Y}$ is on the right.

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 2 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 & 3 & 1 & 0 & 4 \\ 0 & 3 & 4 & 0 & 0 & 1 & 3 & 2 \\ 1 & 3 & 1 & 1 & 2 & 1 & 3 & 0 \end{array}\right) \sim\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 & 3 & 1 & 3 \\ 0 & 0 & 1 & 0 & 3 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 4 & 0 & 0 \end{array}\right)$$

The change of basis matrix from X to Y is: $[id]_{X,Y} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 3 & 1 & 3 \\ 3 & 3 & 0 & 4 \\ 0 & 4 & 0 & 0 \end{pmatrix}$

Characterization of matrices of isomorphisms

Theorem: A linear map $f : U \to V$ is an isomorphism of spaces U and V with finite bases X and Y if and only if $[f]_{X,Y}$ is regular.



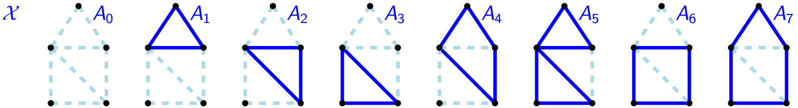
Proof: $\Leftarrow$: Consider $g : V \to U$ such that $[g]_{Y,X} = [f]^{-1}_{X,Y}$. Then:
$[g \circ f]_{X,X} = [f]^{-1}_{X,Y}[f]_{X,Y} = \boldsymbol{I}_{|X|} = [id]_{X,X} \Rightarrow f$ is injective
$[f \circ g]_{Y,Y} = [f]_{X,Y}[f]^{-1}_{X,Y} = \boldsymbol{I}_{|Y|} = [id]_{Y,Y} \Rightarrow f$ is surjective

$$\Rightarrow: \left.\begin{array}{l} [f^{-1}]_{Y,X}[f]_{X,Y} = [id]_{X,X} = \boldsymbol{I}_{|X|} \Rightarrow |Y| \geq |X| \\ [f]_{X,Y}[f^{-1}]_{Y,X} = [id]_{Y,Y} = \boldsymbol{I}_{|Y|} \Rightarrow |X| \geq |Y| \end{array}\right\} \Rightarrow |X| = |Y|$$

Corollary: Any isomorphism f satisfies: $[f^{-1}]_{Y,X} = [f]^{-1}_{X,Y}$.

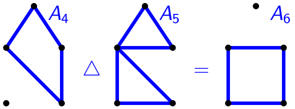
Example of an isomorphism

Let $(\mathcal{X}, \triangle)$ be the vector space of even subgraphs of a graph G. The underlying field is $\mathbb{Z}_2$. The map $f : \mathcal{X} \to \mathbb{Z}_2^3$ given in the table below is linear and bijective, hence an isomorphism.



$f(A_i)$:$(0,0,0)^T$ $(1,0,0)^T$ $(0,1,0)^T$ $(0,0,1)^T$ $(1,1,0)^T$ $(1,0,1)^T$ $(0,1,1)^T$ $(1,1,1)^T$

Linearity holds, e.g:



$(1,1,0)^T + (1,0,1)^T = (0,1,1)^T$

$f(A_4) + f(A_5) = f(A_4 \triangle A_5) = f(A_6)$

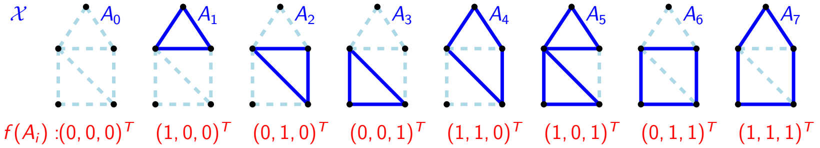
The matrix of the mapping depends on both bases chosen.

E.g. $[f]_{\{A_1,A_2,A_3\},K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The dimension of both spaces is 3.

Use of the matrix

For another choice $X = \{A_4, A_5, A_1\}$ we get $[f]_{X,K} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$



Observe that $[f]_{X,K}[A]_X = [f(A)]_K$ holds.
E.g. for A_6 we get: $A_6 = A_4 \triangle A_5$ and hence $[A_6]_X = (1,1,0)^T$.
Now:

$$[f]_{X,K}[A_6]_X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = [f(A_6)]_K$$