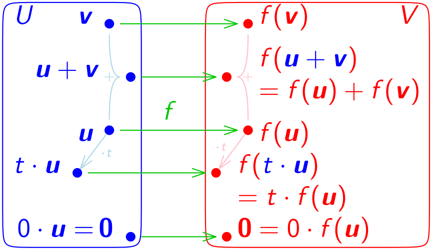
Linear mapping

Observation: Let $\boldsymbol{A} \in F^{m\times n}$ and
$f : F^n \to F^m$ be defined as $f(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$. Then:

- $f(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{u}+\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{A}\boldsymbol{v} = f(\boldsymbol{u}) + f(\boldsymbol{v})$
- $f(t\cdot\boldsymbol{u}) = \boldsymbol{A}(t\cdot\boldsymbol{u}) = t\cdot\boldsymbol{A}\boldsymbol{u} = t\cdot f(\boldsymbol{u})$

Definition: Let U and V be vector spaces over the same field F.
A mapping $f : U \to V$ is a *linear mapping* if:

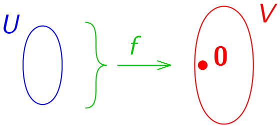
- $\forall \boldsymbol{u}, \boldsymbol{v} \in U$:
  $f(\boldsymbol{u}+\boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$
- $\forall \boldsymbol{u} \in U, \forall t \in F$:
  $f(t\cdot\boldsymbol{u}) = t\cdot f(\boldsymbol{u})$



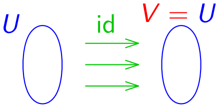
Observation: Each linear mapping satisfies: $f(\mathbf{0}) = \mathbf{0}$.

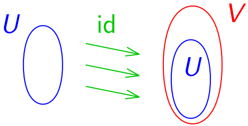
Examples of simple linear mappings

Between general vector spaces $f : U \to V$ over the same F.
The *trivial* linear mapping given by: $\forall \boldsymbol{u} \in U : f(\boldsymbol{u}) = \mathbf{0}$.



The *identity* id on U given by: $\forall \boldsymbol{u} \in U : \mathrm{id}(\boldsymbol{u}) = \boldsymbol{u}$.



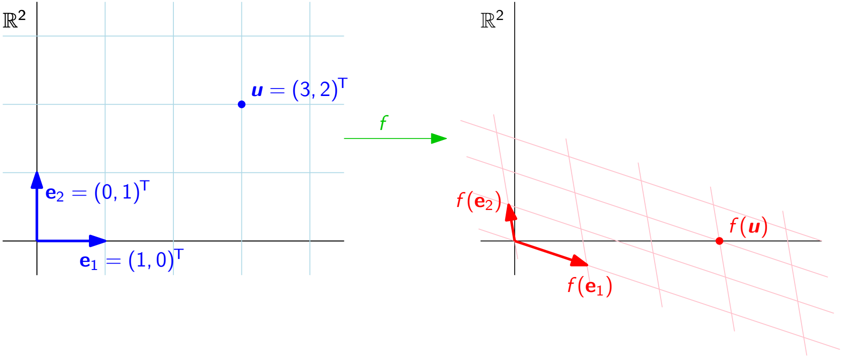


The identity is also an immersion of U into V when $U \subseteq V$.
Linearity of both addition and scalar multiplication is here obvious.

Geometric linear mappings

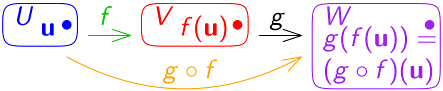
Some geometric transformations in $\mathbb{R}^2$ or $\mathbb{R}^3$ that fix the origin:

- rotation around the origin
- reflection across an axis that goes through the origin
- scaling with the center in the origin,
  including non-uniform scaling and projection
- any similarity transformation that combines the above



Properties of linear mappings

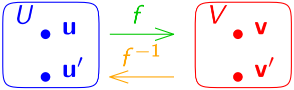
Observation: If $f : U \to V$, $g : V \to W$ are linear maps, then $(g \circ f) : U \to W$ is linear.



Proof: $(g \circ f)(\boldsymbol{u} + \boldsymbol{v}) = g(f(\boldsymbol{u} + \boldsymbol{v})) = g(f(\boldsymbol{u}) + f(\boldsymbol{v})) =$
$= g(f(\boldsymbol{u})) + g(f(\boldsymbol{v})) = (g \circ f)(\boldsymbol{u}) + (g \circ f)(\boldsymbol{v})$
$(g \circ f)(t\boldsymbol{u}) = g(f(t\boldsymbol{u})) = g(tf(\boldsymbol{u})) = tg(f(\boldsymbol{u})) = t(g \circ f)(\boldsymbol{u})$

Observation: If $f : U \to V$ is a linear bijective mapping, then $f^{-1} : V \to U$ is a linear map too.

Proof: For any $\boldsymbol{v}, \boldsymbol{v}' \in V$
let $\boldsymbol{u} = f^{-1}(\boldsymbol{v})$ and $\boldsymbol{u}' = f^{-1}(\boldsymbol{v}')$,
that is, $f(\boldsymbol{u}) = \boldsymbol{v}$ and $f(\boldsymbol{u}') = \boldsymbol{v}'$.



Linearity of addition: $f^{-1}(\boldsymbol{v} + \boldsymbol{v}') = f^{-1}(f(\boldsymbol{u}) + f(\boldsymbol{u}')) =$
$f^{-1}(f(\boldsymbol{u} + \boldsymbol{u}')) = \boldsymbol{u} + \boldsymbol{u}' = f^{-1}(\boldsymbol{v}) + f^{-1}(\boldsymbol{v}')$

Linearity of scalar multiplication:
$\forall t \in F : f^{-1}(t\boldsymbol{v}) = f^{-1}(tf(\boldsymbol{u})) = f^{-1}(f(t\boldsymbol{u})) = t\boldsymbol{u} = tf^{-1}(\boldsymbol{v})$.

Definition: A bijective linear mapping is called an *isomorphism*.

Transformation to the vector of coordinates

Proposition: For a space U over F with a basis $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ the mapping $f : U \to F^n$ defined as $f(\boldsymbol{u}) = [\boldsymbol{u}]_B$ is linear.

Proof: For $\boldsymbol{u}, \boldsymbol{v} \in U$: express $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i$, $\boldsymbol{v} = \sum_{i=1}^{n} c_i \boldsymbol{b}_i$, i.e. the coordinate vectors are $[\boldsymbol{u}]_B = (a_1, \ldots, a_n)^\mathsf{T}$, $[\boldsymbol{v}]_B = (c_1, \ldots, c_n)^\mathsf{T}$.

L. of addition: $f(\boldsymbol{u} + \boldsymbol{v}) = [\boldsymbol{u} + \boldsymbol{v}]_B = \left[\sum_{i=1}^{n} a_i \boldsymbol{b}_i + \sum_{i=1}^{n} c_i \boldsymbol{b}_i\right]_B =$

$= \left[\sum_{i=1}^{n} (a_i + c_i) \boldsymbol{b}_i\right]_B = (a_1 + c_1, \ldots, a_n + c_n)^\mathsf{T} =$

$= (a_1, \ldots, a_n)^\mathsf{T} + (c_1, \ldots, c_n)^\mathsf{T} = [\boldsymbol{u}]_B + [\boldsymbol{v}]_B = f(\boldsymbol{u}) + f(\boldsymbol{v})$

Linearity of scalar multiplication:

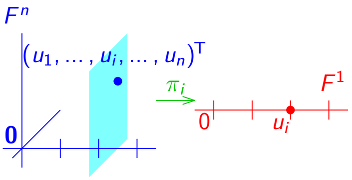
For $t \in F : f(t\boldsymbol{u}) = [t\boldsymbol{u}]_B = \left[t \sum_{i=1}^{n} a_i \boldsymbol{b}_i\right]_B = \left[\sum_{i=1}^{n} t a_i \boldsymbol{b}_i\right]_B =$

$= (ta_1, \ldots, ta_n)^\mathsf{T} = t(a_1, \ldots, a_n)^\mathsf{T} = t[\boldsymbol{u}]_B = tf(\boldsymbol{u})$

Observation: The map $\boldsymbol{u} \leftrightarrow [\boldsymbol{u}]_B$ is a bijection, i.e. an isomorphism.

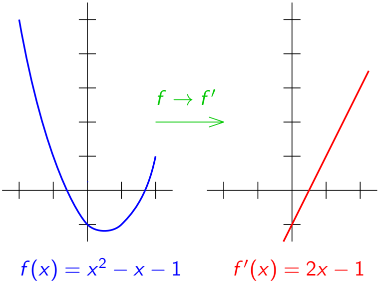
Further examples of linear maps

In arithmetic vector spaces the *projection* to the i-th coordinate, i.e. $\pi_i : F^n \to F^1$ given $\pi_i((u_1, \dots, u_n)^\mathsf{T}) = u_i$, is a linear mapping.



Note: We only write u_i instead formally correct $(u_i)^\mathsf{T}$.

On the space of functions with derivatives of all degrees

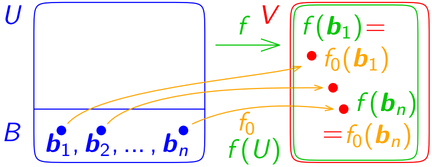


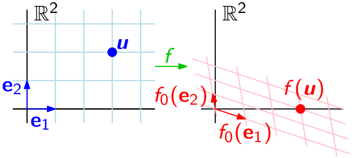
differentiation is a linear map:

$$(f(x) + g(x))' = f'(x) + g'(x)$$
$$(t \cdot f(x))' = t \cdot f'(x)$$

Extension theorem

Theorem: Let U and V be spaces over F and B be a basis of U. Then for any mapping $f_0 : B \to V$ there exists a unique linear map $f : U \to V$ that extends f_0, i.e. $\forall \boldsymbol{b} \in B : f(\boldsymbol{b}) = f_0(\boldsymbol{b})$.





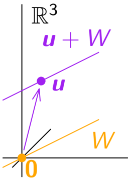
Proof: For any $\boldsymbol{u} \in U$ there are unique $n \in \mathbb{N}_0$, $a_1, \ldots, a_n \in F \setminus 0$ and $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n \in B$ such that $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i$. Then $f(\boldsymbol{u})$ is uniquely determined by $f(\boldsymbol{u}) = f\left(\sum_{i=1}^{n} a_i \boldsymbol{b}_i\right) = \sum_{i=1}^{n} a_i f(\boldsymbol{b}_i) = \sum_{i=1}^{n} a_i f_0(\boldsymbol{b}_i)$.

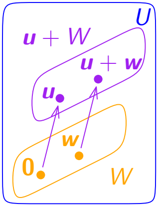
Corollary: If $f : U \to V$ is linear then $\dim(U) \geq \dim(f(U))$, because the image $f(B)$ of a basis B of U generates $f(U)$.

Affine spaces

Definition: Let W be a subspace of a vector space U and $\boldsymbol{u} \in U$.
The *affine subspace* $\boldsymbol{u} + W$ is the set $\{\boldsymbol{u} + \boldsymbol{w} : \boldsymbol{w} \in W\}$.
The *dimension* of the affine space $\boldsymbol{u} + W$ is the value $\dim(W)$.

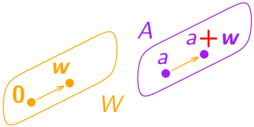
Example: Lines, planes (hyperplanes) in general position in $\mathbb{R}^3$ (in $\mathbb{R}^d$).





Note: An affine space can be defined more generally as a set A together with a mapping $+ \colon A \times W \to A$ satisfying:

- $\forall a \in A, \forall \boldsymbol{v}, \boldsymbol{w} \in W :$
  $a + (\boldsymbol{v} + \boldsymbol{w}) = (a + \boldsymbol{v}) + \boldsymbol{w}$
- $\forall a, b \in A \ \exists! \boldsymbol{v} \in W : a + \boldsymbol{v} = b$



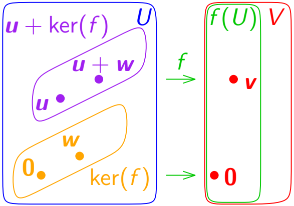
Elements of A are called *points* (neither scalars nor vectors).

Observation: For every $a \in A$, the following holds: $a + \mathbf{0} = a$.

Proof: Let $\boldsymbol{v}$ be the unique vector satisfying $a + \boldsymbol{v} = a$, then
$a = a + \boldsymbol{v} = (a + \boldsymbol{v}) + \boldsymbol{v} = a + (\boldsymbol{v} + \boldsymbol{v}) \ \Rightarrow \ \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{v} \ \Rightarrow \ \boldsymbol{v} = \mathbf{0}$.

The preimage of a vector in a linear mapping

Definition:
The *kernel* of a linear mapping $f : U \to V$ is $\ker(f) = \{\boldsymbol{w} \in U : f(\boldsymbol{w}) = \boldsymbol{0}\}$.



Observation: Kernel is a *vector* subspace.

Observation: For $f : F^n \to F^m$ given by $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ we get $\ker(f) = \ker(\boldsymbol{A})$.

Theorem: Let $f : U \to V$ be a linear mapping. For any $\boldsymbol{v} \in V$ the equation $f(\boldsymbol{x}) = \boldsymbol{v}$ has either no solution or the solutions form an affine subspace $\boldsymbol{u} + \ker(f)$, where $\boldsymbol{u}$ is any solution of $f(\boldsymbol{x}) = \boldsymbol{v}$.

Examples: Solutions of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$; the constant $+c$ in integration.

Proof: When $\boldsymbol{x} \in \boldsymbol{u} + \ker(f)$ then $\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{w}$ with $\boldsymbol{w} \in \ker(f)$.
Now $f(\boldsymbol{x}) = f(\boldsymbol{u} + \boldsymbol{w}) = f(\boldsymbol{u}) + f(\boldsymbol{w}) = \boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$.

Conversely, $f(\boldsymbol{x}) = \boldsymbol{v} \;\Rightarrow\; f(\boldsymbol{x} - \boldsymbol{u}) = f(\boldsymbol{x}) - f(\boldsymbol{u}) = \boldsymbol{v} - \boldsymbol{v} = \boldsymbol{0}$,
thus $\boldsymbol{x} - \boldsymbol{u} \in \ker(f)$, and therefore $\boldsymbol{x} \in \boldsymbol{u} + \ker(f)$.

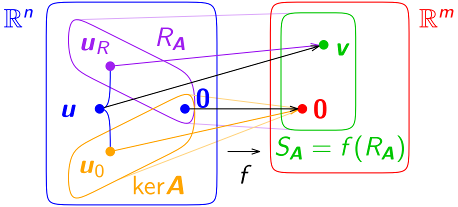
Bonus: alternative proof of $\dim R_{\boldsymbol{A}} = \dim C_{\boldsymbol{A}}$ over $\mathbb{R}$

For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ consider $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f(\boldsymbol{x}) = \boldsymbol{Ax}$.
We know that $C_{\boldsymbol{A}} = f(\mathbb{R}^n)$. We indeed show that $C_{\boldsymbol{A}} = f(R_{\boldsymbol{A}})$.

For every $\boldsymbol{v} \in C_{\boldsymbol{A}}$, there exists $\boldsymbol{u} \in \mathbb{R}^n$ such that $f(\boldsymbol{u}) = \boldsymbol{v}$.

Since $\dim R_{\boldsymbol{A}} + \dim(\ker \boldsymbol{A}) = n$ and $R_{\boldsymbol{A}} \cap \ker \boldsymbol{A} = \{\mathbf{0}\}$, the union of the bases $R_{\boldsymbol{A}}$ and $\ker \boldsymbol{A}$ is a basis for $\mathbb{R}^n$. The vector $\boldsymbol{u}$ can be written as $\boldsymbol{u} = \boldsymbol{u}_R + \boldsymbol{u}_0$ with $\boldsymbol{u}_R \in R_{\boldsymbol{A}}$ and $\boldsymbol{u}_0 \in \ker \boldsymbol{A}$. We get:

$$\boldsymbol{v} = f(\boldsymbol{u}) = f(\boldsymbol{u}_R + \boldsymbol{u}_0) = f(\boldsymbol{u}_R) + f(\boldsymbol{u}_0) = f(\boldsymbol{u}_R) + \mathbf{0} = f(\boldsymbol{u}_R)$$



From $f(R_{\boldsymbol{A}}) = C_{\boldsymbol{A}}$ it follows that $\dim R_{\boldsymbol{A}} \geq \dim C_{\boldsymbol{A}}$. Analogously, $f(R_{\boldsymbol{A}^\top}) = C_{\boldsymbol{A}^\top}$ yields $\dim C_{\boldsymbol{A}} = \dim R_{\boldsymbol{A}^\top} \geq \dim C_{\boldsymbol{A}^\top} = \dim R_{\boldsymbol{A}}$.

Questions to understand the lecture topic

- For which of the axioms of linear mapping is it necessary that both spaces be over the same field?
- If $f(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$, $g(\boldsymbol{u}) = \boldsymbol{B}\boldsymbol{u}$, and they can be composed, what does the mapping $g \circ f$ correspond to?
- Which of the mappings in the examples are isomorphisms and which are not?
- Why is it necessary for the extension theorem to require the uniqueness of n and the coefficients a_i to be non-zero?
- What properties would the mapping f_0 from the extension theorem have to have in order for f to be injective, or to be an isomorphism?
- Why is the image $f(U)$ of the space U a subspace of V and not just a subset?

Questions to understand the lecture topic

- Is a linear mapping uniquely determined by the image of a linearly independent set or by the image of a system of generators?
- Note that images of linear mappings can be added and multiplied by a scalar, thus defining their sum and scalar multiple. What algebraic structure does the set of all linear mappings $\{f : U \to V\}$ with these two operations form?
- What is the geometric interpretation of affine spaces in $\mathbb{R}^3$?
- Can two different affine spaces of the same dimension have a nonempty intersection? Is this possible even if they are determined by the same space W?