Linear maps

Observation: Let $\boldsymbol{A} \in \mathbb{K}^{m \times n}$ and $f : \mathbb{K}^n \to \mathbb{K}^m$ be defined as $f(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u}$. Then:

- $f(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{A}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{A}\boldsymbol{v} = f(\boldsymbol{u}) + f(\boldsymbol{v})$
- $f(\alpha \boldsymbol{u}) = \boldsymbol{A}(\alpha \boldsymbol{u}) = \alpha \boldsymbol{A}\boldsymbol{u} = \alpha f(\boldsymbol{u})$

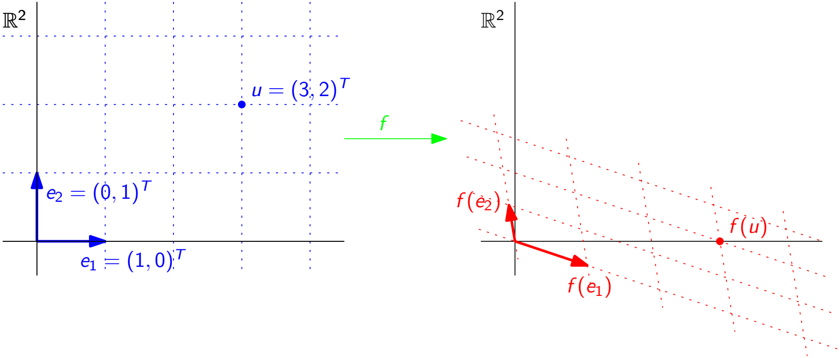
Definition: Let U and V be vector spaces over the same field $\mathbb{K}$. A mapping $f : U \to V$ is a *linear map* if:

- $\forall \boldsymbol{u}, \boldsymbol{v} \in U : f(\boldsymbol{u} + \boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})$
- $\forall \boldsymbol{u} \in U, \forall \alpha \in \mathbb{K} : f(\alpha \boldsymbol{u}) = \alpha f(\boldsymbol{u})$

Examples of linear maps

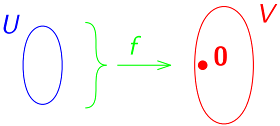
Some geometric transformations in $\mathbb{R}^2$ or $\mathbb{R}^3$ that fix the origin:

- rotation around the origin
- reflection across an axis that goes through the origin
- scaling with the center in the origin, including non-uniform scaling and projection
- any similarity transformation that combines the above

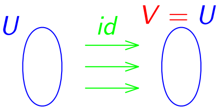


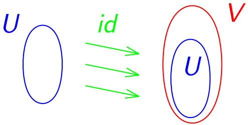
Among general vector spaces $f : U \to V$

The *trivial* linear map: $\forall \boldsymbol{u} \in U : f(\boldsymbol{u}) = \mathbf{0}$.



The *identity* id on U





or as an immersion of U into V when $U \subseteq V$.

Transformation into the vector of coordinates

Let U be a space over $\mathbb{K}$ with a basis $X = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$.

The map $f : U \to \mathbb{K}^n$ defined as $f(\boldsymbol{u}) = [\boldsymbol{u}]_X$ is linear.

For $\boldsymbol{w}, \boldsymbol{w}' \in U$: express $\boldsymbol{w} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i$ and $\boldsymbol{w}' = \sum_{i=1}^{n} b_i \boldsymbol{u}_i$,
i.e. $[\boldsymbol{w}]_X = (a_1, \ldots, a_n)^T$, $[\boldsymbol{w}']_X = (b_1, \ldots, b_n)^T$.

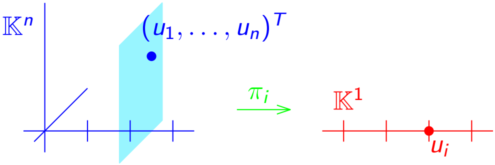
$$f(\boldsymbol{w} + \boldsymbol{w}') = [\boldsymbol{w} + \boldsymbol{w}']_X = \left[\sum_{i=1}^{n} a_i \boldsymbol{u}_i + \sum_{i=1}^{n} b_i \boldsymbol{u}_i\right]_X =$$
$$= \left[\sum_{i=1}^{n} (a_i + b_i)\boldsymbol{u}_i\right]_X = (a_1 + b_1, \ldots, a_n + b_n)^T =$$
$$= (a_1, \ldots, a_n)^T + (b_1, \ldots, b_n)^T = [\boldsymbol{w}]_X + [\boldsymbol{w}']_X = f(\boldsymbol{w}) + f(\boldsymbol{w}')$$

For $\alpha \in \mathbb{K} : f(\alpha \boldsymbol{w}) = [\alpha \boldsymbol{w}]_X = \left[\alpha \sum_{i=1}^{n} a_i \boldsymbol{u}_i\right]_X = \left[\sum_{i=1}^{n} \alpha a_i \boldsymbol{u}_i\right]_X =$
$$= (\alpha a_1, \ldots, \alpha a_n)^T = \alpha(a_1, \ldots, a_n)^T = \alpha[\boldsymbol{w}]_X = \alpha f(\boldsymbol{w})$$

The projection

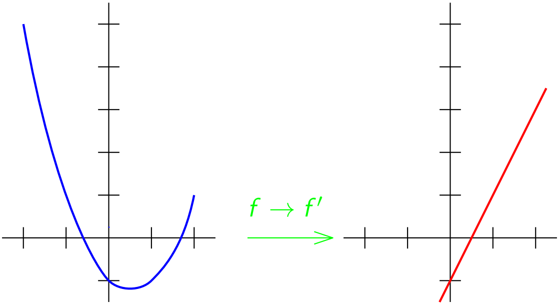
On an arithmetic vector space, the *projection* to the i-th coordinate

$\pi_i : \mathbb{K}^n \to \mathbb{K}^1$: $\pi_i((u_1, \ldots, u_n)^T) = u_i$.



Differentiation

On the space of functions with derivatives of all degrees



$f(x) = x^2 - x - 1$ $\qquad$ $f'(x) = 2x - 1$

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$(\alpha \cdot f(x))' = \alpha \cdot f'(x)$$

Properties of linear maps

Observation: If $f : U \to V$ and $g : V \to W$ are linear maps, then $(g \circ f) : U \to W$ is also linear.

Proof: $(g \circ f)(\boldsymbol{u} + \boldsymbol{v}) = g(f(\boldsymbol{u} + \boldsymbol{v})) = g(f(\boldsymbol{u}) + f(\boldsymbol{v})) =$
$= g(f(\boldsymbol{u})) + g(f(\boldsymbol{v})) = (g \circ f)(\boldsymbol{u}) + (g \circ f)(\boldsymbol{v})$
$(g \circ f)(\alpha \boldsymbol{u}) = g(f(\alpha \boldsymbol{u})) = g(\alpha f(\boldsymbol{u})) = \alpha g(f(\boldsymbol{u})) = \alpha (g \circ f)(\boldsymbol{u})$

Theorem: Let U and V be spaces over $\mathbb{K}$ and X be a basis of U. Then for any mapping $f_0 : X \to V$ there exists a unique linear map $f : U \to V$ that extends f_0, i.e. $\forall \boldsymbol{u} \in X : f(\boldsymbol{u}) = f_0(\boldsymbol{u})$.

Proof: For any $\boldsymbol{w} \in U$ there are unique $n \in \mathbb{N}_0$, $a_1, \ldots, a_n \in K \setminus 0$ and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in X$ such that $\boldsymbol{w} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i$. Then $f(\boldsymbol{w})$ is uniquely determined by $f(\boldsymbol{w}) = f\left(\sum_{i=1}^{n} a_i \boldsymbol{u}_i\right) = \sum_{i=1}^{n} a_i f(\boldsymbol{u}_i) = \sum_{i=1}^{n} a_i f_0(\boldsymbol{u}_i)$.

Corollary: If $f : U \to V$ is linear then $\dim(U) \geq \dim(f(U))$, because the image $f(X)$ of a basis X of U generates $f(U)$.

Affine subspaces

Definition: Let W be a subspace of a vector space U and $\boldsymbol{u} \in U$. The *affine subspace* $\boldsymbol{u} + W$ is the set $\{\boldsymbol{u} + \boldsymbol{w} : \boldsymbol{w} \in W\}$. The *dimension* of an affine space $\boldsymbol{u} + W$ is $\dim(\boldsymbol{u} + W) = \dim(W)$.

Example: Lines, planes and hyperplanes in general position in $\mathbb{R}^d$.

Definition: The *kernel* of a linear map $f : U \to V$ is $\ker(f) = \{u \in U : f(u) = \mathbf{0}\}$.

Observation: Each kernel is a subspace.

Theorem: Let $f : U \to V$ be a linear map. For any $\boldsymbol{v} \in V$ the equation $f(\boldsymbol{u}) = \boldsymbol{v}$ has either no solution or the solutions form an affine subspace $\boldsymbol{u}_0 + \ker(f)$, where $\boldsymbol{u}_0$ is any solution of $f(\boldsymbol{u}) = \boldsymbol{v}$.

Proof: When $\boldsymbol{u} \in \boldsymbol{u}_0 + \ker(f)$ then $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$ with $\boldsymbol{w} \in \ker(f)$. Now $f(\boldsymbol{u}) = f(\boldsymbol{u}_0 + \boldsymbol{w}) = f(\boldsymbol{u}_0) + f(\boldsymbol{w}) = \boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$.

If $f(\boldsymbol{u}) = \boldsymbol{v}$ then $f(\boldsymbol{u} - \boldsymbol{u}_0) = f(\boldsymbol{u}) - f(\boldsymbol{u}_0) = \boldsymbol{v} - \boldsymbol{v} = \mathbf{0}$, i.e. $\boldsymbol{u} - \boldsymbol{u}_0 \in \ker(f)$, hence $\boldsymbol{u} \in \boldsymbol{u}_0 + \ker(f)$.