Spaces determined by a matrix $\boldsymbol{A} \in F^{m \times n}$

Definition: *Kernel* is the set of solutions of $\boldsymbol{Ax} = \mathbf{0}$, denoted $\ker \boldsymbol{A}$, *column space* is the subspace of F^m generated by columns of $\boldsymbol{A}$, *row space* is generated in F^n by transposes of rows of $\boldsymbol{A}$.

Example: For the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \in \mathbb{Z}_3^{3\times 4}$ we get

The row space:

$$R_{\boldsymbol{A}} = \left\{ \begin{matrix} (0,0,0,0)^{\mathsf{T}}, & (1,2,0,1)^{\mathsf{T}}, & (2,1,0,2)^{\mathsf{T}}, \\ (2,0,2,1)^{\mathsf{T}}, & (0,2,2,2)^{\mathsf{T}}, & (1,1,2,0)^{\mathsf{T}}, \\ (1,0,1,2)^{\mathsf{T}}, & (2,2,1,0)^{\mathsf{T}}, & (0,1,1,1)^{\mathsf{T}} \end{matrix} \right\} \subseteq \mathbb{Z}_3^4$$

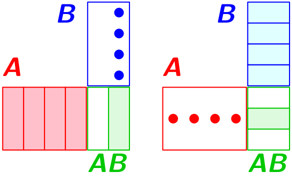
The column space:

$$C_{\boldsymbol{A}} = \left\{ \begin{matrix} (0,0,0)^{\mathsf{T}}, & (1,2,1)^{\mathsf{T}}, & (2,1,2)^{\mathsf{T}}, \\ (2,0,1)^{\mathsf{T}}, & (0,2,2)^{\mathsf{T}}, & (1,1,0)^{\mathsf{T}}, \\ (1,0,2)^{\mathsf{T}}, & (2,2,0)^{\mathsf{T}}, & (0,1,1)^{\mathsf{T}} \end{matrix} \right\} \subseteq \mathbb{Z}_3^3$$

Denote row and column spaces of $\boldsymbol{A}$ by the symbols $R_{\boldsymbol{A}}$ and $C_{\boldsymbol{A}}$.

Properties

Formally: The column space is
$C_{\boldsymbol{A}} = \{\boldsymbol{Ac} : \boldsymbol{c} \in F^n\} =$
$= \{\boldsymbol{u} \in F^m : \exists \boldsymbol{c} \in F^n : \boldsymbol{u} = \boldsymbol{Ac}\}$,
and similarly, the row space is
$R_{\boldsymbol{A}} = \{\boldsymbol{v} \in F^n : \exists \boldsymbol{d} \in F^m : \boldsymbol{v} = \boldsymbol{A}^{\mathsf{T}}\boldsymbol{d}\}$.



Observation: The system $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution iff $\boldsymbol{b} \in C_{\boldsymbol{A}}$.

Observation: The kernel $\ker \boldsymbol{A} = \{\boldsymbol{x} \in F^n : \boldsymbol{Ax} = \mathbf{0}\}$ is a subspace.

Observation: Elementary transforms do not alter $R_{\boldsymbol{A}}$ nor $\ker \boldsymbol{A}$.
... from the systems of equations and the exchange lemma.

Theorem: For any $\boldsymbol{A} \in F^{m \times n} : \dim(\ker \boldsymbol{A}) + \operatorname{rank} \boldsymbol{A} = n$

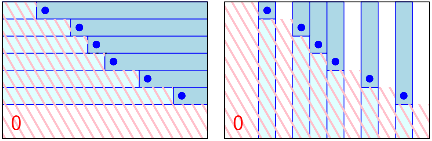
Proof: Let $d = n - \operatorname{rank}(\boldsymbol{A})$ be the number of free variables and $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ solutions of $\boldsymbol{Ax} = \mathbf{0}$ obtained by backward substitution.

These $\boldsymbol{x}_1, \dots, \boldsymbol{x}_d$ are linearly independent, as each $\boldsymbol{x}_i$ is unique with the coordinate corresponding to the i-th free variable nonzero.

A basis of $\ker \boldsymbol{A}$ is $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_d\}$ and $\dim(\ker \boldsymbol{A}) = d = n - \operatorname{rank}(\boldsymbol{A})$.

Dimensions of row and column spaces coincide

Observation: For every $\boldsymbol{A}'$ in echelon form:
$\dim R_{\boldsymbol{A}'} = \dim C_{\boldsymbol{A}'} = \operatorname{rank} \boldsymbol{A}'$



Lemma: If $\boldsymbol{A} \sim\sim \boldsymbol{A}'$, where $\boldsymbol{A}'$ is in echelon form, then the columns of $\boldsymbol{A}$ corresponding to the leading variables form a basis $C_{\boldsymbol{A}}$.

Proof: Denote $\operatorname{rank} \boldsymbol{A} = r$, the indices of the leading variables $j(1), \ldots, j(r)$ and the related columns of $\boldsymbol{A}$ by $\boldsymbol{c}_{j(1)}, \ldots, \boldsymbol{c}_{j(r)}$.
For every $\boldsymbol{b} \in C_{\boldsymbol{A}}$, the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x}$, in which all free variables have value 0.
From $\boldsymbol{b} = \sum_{i=1}^{r} x_{j(i)} \boldsymbol{c}_{j(i)}$ follows that $\operatorname{span}(\{\boldsymbol{c}_{j(1)}, \ldots, \boldsymbol{c}_{j(r)}\}) = C_{\boldsymbol{A}}$.
Vectors $\boldsymbol{c}_{j(1)}, \ldots, \boldsymbol{c}_{j(r)}$ are linearly independent, as only $\boldsymbol{x} = \mathbf{0}$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$, where all free variables have value 0.

Theorem: For each $\boldsymbol{A}$: $\dim R_{\boldsymbol{A}} = \dim C_{\boldsymbol{A}}$, thus $\operatorname{rank} \boldsymbol{A} = \operatorname{rank}(\boldsymbol{A}^{\mathsf{T}})$.

Proof: $\dim R_{\boldsymbol{A}} = \dim R_{\boldsymbol{A}'} = \dim C_{\boldsymbol{A}'} = \operatorname{rank} \boldsymbol{A}' = \operatorname{rank} \boldsymbol{A} = \dim C_{\boldsymbol{A}}$.

Relationship to the matrix product

Observation: For column spaces of $\boldsymbol{A}$ and $\boldsymbol{AB}$: $C_{\boldsymbol{AB}} \subseteq C_{\boldsymbol{A}}$.

Proof: For $\boldsymbol{A} \in F^{m\times n}$ and $\boldsymbol{B} \in F^{n\times p}$, it comes from the definition: $\{\boldsymbol{ABc} : \boldsymbol{c} \in F^p\} \subseteq \{\boldsymbol{Ad} : \boldsymbol{d} \in F^n\}$, since $\{\boldsymbol{Bc} : \boldsymbol{c} \in F^p\} \subseteq F^n$.

Observation: Analogously for row spaces: $R_{\boldsymbol{AB}} \subseteq R_{\boldsymbol{B}}$.

Consequence: $\operatorname{rank}(\boldsymbol{AB}) \leq \min\{\operatorname{rank}\boldsymbol{A}, \operatorname{rank}\boldsymbol{B}\}$

Beware! Column spaces $\boldsymbol{A}$ and $\boldsymbol{BA}$ satisfy only: $\dim C_{\boldsymbol{BA}} \leq \dim C_{\boldsymbol{A}}$.

Example: For the product $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

we have: $\dim C_{\boldsymbol{BA}} = 1 \leq 2 = \dim C_{\boldsymbol{A}}$. The matrix $\boldsymbol{B}$ is purposely singular in order to decrease $\dim C_{\boldsymbol{BA}}$ when compared to $\dim C_{\boldsymbol{A}}$.

Theorem: For any matrix $\boldsymbol{A} \in F^{m\times n}$, regular $\boldsymbol{R} \in F^{m\times m}$ and regular $\boldsymbol{R}' \in F^{n\times n}$: $\operatorname{rank}\boldsymbol{A} = \operatorname{rank}(\boldsymbol{RA}) = \operatorname{rank}(\boldsymbol{AR}')$.

Proof: $\operatorname{rank}\boldsymbol{A} \geq \operatorname{rank}(\boldsymbol{RA}) \geq \operatorname{rank}(\boldsymbol{R}^{-1}\boldsymbol{RA}) = \operatorname{rank}\boldsymbol{A}$; for $\boldsymbol{R}'$ too.

Relationships between row space and kernel

Observations:

- For every $\boldsymbol{v} \in R_{\boldsymbol{A}}$ and every $\boldsymbol{x} \in \ker \boldsymbol{A}$: $\boldsymbol{v}^\mathsf{T}\boldsymbol{x} = 0$,
- for every $\boldsymbol{x} \in F^n$: $(\forall \boldsymbol{v} \in R_{\boldsymbol{A}}\colon \boldsymbol{v}^\mathsf{T}\boldsymbol{x} = 0) \Leftrightarrow \boldsymbol{x} \in \ker \boldsymbol{A}$,
- for every $\boldsymbol{v} \in F^n$: $(\forall \boldsymbol{x} \in \ker \boldsymbol{A}\colon \boldsymbol{v}^\mathsf{T}\boldsymbol{x} = 0) \Leftrightarrow \boldsymbol{v} \in R_{\boldsymbol{A}}$.

Proof: We choose a suitable $\boldsymbol{d} \in F^m$ such that $\boldsymbol{v} = \boldsymbol{A}^\mathsf{T}\boldsymbol{d}$.
Then it holds that: $\boldsymbol{v}^\mathsf{T}\boldsymbol{x} = (\boldsymbol{A}^\mathsf{T}\boldsymbol{d})^\mathsf{T}\boldsymbol{x} = \boldsymbol{d}^\mathsf{T}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{d}^\mathsf{T}\mathbf{0} = 0$.
The first comes from the first observation and kernel definition.
The second follows from the fact that adding $\boldsymbol{v} \notin R_{\boldsymbol{A}}$ as a new row to $\boldsymbol{A}$ increase rank. Some free variable become leading, which provides a construction of $\boldsymbol{x} \in \ker \boldsymbol{A}$ such that $\boldsymbol{v}^\mathsf{T}\boldsymbol{x} \neq 0$.

Observation: For *real* matrices $\boldsymbol{A}$, we have $R_{\boldsymbol{A}} \cap \ker \boldsymbol{A} = \{\mathbf{0}\}$.

Proof: Choose $\boldsymbol{v} \in R_{\boldsymbol{A}} \cap \ker \boldsymbol{A}$. Since $\boldsymbol{v} \in R_{\boldsymbol{A}}$, the equation $v_1x_1 + \cdots + v_nx_n = 0$ is a linear combination of rows of $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$.
Since $\boldsymbol{v} \in \ker \boldsymbol{A}$, the vector $\boldsymbol{v}$ solves this equation. Substitute it to obtain: $v_1v_1 + \cdots + v_nv_n = 0$, from which on $\mathbb{R}$ follows that $\boldsymbol{v} = \mathbf{0}$.

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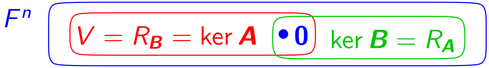
- for every $\boldsymbol{v} \in F^n$: $(\forall \boldsymbol{x} \in \ker \boldsymbol{A}\colon \boldsymbol{v}^{\mathsf{T}}\boldsymbol{x} = 0) \Leftrightarrow \boldsymbol{v} \in R_{\boldsymbol{A}}$.

Theorem: Every subspace V of the arithmetic vector space F^n is a set of solutions to the appropriate homogeneous system $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$.

Proof: From the basis V, we construct row-wise the auxiliary matrix $\boldsymbol{B}$. Similarly, from the basis $\ker \boldsymbol{B}$, we construct row-wise the desired matrix $\boldsymbol{A}$.



$$\boldsymbol{x} \in V \Leftrightarrow \boldsymbol{x} \in R_{\boldsymbol{B}} \Leftrightarrow \forall \boldsymbol{v} \in \ker \boldsymbol{B}\colon \boldsymbol{v}^{\mathsf{T}}\boldsymbol{x} = 0 \Leftrightarrow$$
$$\Leftrightarrow \forall \boldsymbol{v} \in R_{\boldsymbol{A}}\colon \quad \boldsymbol{v}^{\mathsf{T}}\boldsymbol{x} = 0 \Leftrightarrow \boldsymbol{x} \in \ker \boldsymbol{A} \Leftrightarrow \boldsymbol{A}\boldsymbol{x} = \mathbf{0}$$

Questions to understand the lecture topic

- How would you describe vectors from the column space of a matrix in row echelon form?
- Does this space change if the matrix is further transformed to reduced row echelon form?
- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices with the same number of columns, how can they be combined to form a matrix with kernel $\ker(\boldsymbol{A}) \cap \ker(\boldsymbol{B})$?
- Can the following hold for appropriately chosen matrices $\operatorname{rank}(\boldsymbol{BA}) < \min\{\operatorname{rank} \boldsymbol{A}, \operatorname{rank} \boldsymbol{B}\}$?
  If so, can you construct such matrices?
- How do you construct $\boldsymbol{x} \in \ker \boldsymbol{A}$ for $\boldsymbol{v} \notin R_A$ so that $\boldsymbol{v}^\mathsf{T} \boldsymbol{x} \neq 0$?