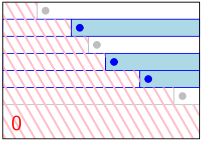
Linear independence

Definition: A set of vectors B is *linearly independent*, if for any n-tuple of vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n \in B$ the equation $\sum_{i=1}^{n} a_i \boldsymbol{b}_i = \mathbf{0}$ has only trivial solution $a_1 = \cdots = a_n = 0$.
In other cases the set B is *linearly dependent*.

Observation: If $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ are linearly dependent, then $\sum_{i=1}^{n} a_i \boldsymbol{b}_i = \mathbf{0}$, where some $a_i \neq 0$. Hence the corresponding $\boldsymbol{b}_i$ can be expressed as a linear combination of the remaining vectors: $\boldsymbol{b}_i = \sum_{j \neq i} -\frac{a_j}{a_i} \boldsymbol{b}_j$.

Examples

- When $\mathbf{0} \in B$ then B is linearly dependent as $1 \cdot \mathbf{0} = \mathbf{0}$ is a nontrivial linear combination.
- Rows or columns of $\mathbf{I}$ are linearly independent.
- Rows of a matrix in row echelon form are linearly independent.
  ... a pivot cannot be eliminated by the zeros below.



- In $\mathbb{R}^2$: $B = \{\boldsymbol{b}\}$ is linearly independent iff $\boldsymbol{b} \neq \mathbf{0}$;
  The set $C = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ is linearly independent iff the line determined by $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ does not contain the origin.
  Any D of size at least three is linearly dependent.
- In the vector space of real polynomials, the infinite set $\{x^0, x^1, x^2, \dots\}$ is linearly independent.
- The empty set is linearly independent.

Two distinct tests of linear independence in F^n

Is $B = \{(2,1,0,3)^\mathsf{T}, (4,3,1,4)^\mathsf{T}, (0,2,2,1)^\mathsf{T}, (3,4,1,0)^\mathsf{T}, (0,2,2,2)^\mathsf{T}\}$ linearly dependent or independent set in $\mathbb{Z}_5^4$?

a) As elementary operations do not modify the *row space*:

$$\begin{pmatrix} 2 & 1 & 0 & 3 \\ 4 & 3 & 1 & 4 \\ 0 & 2 & 2 & 1 \\ 3 & 4 & 1 & 0 \\ 0 & 2 & 2 & 2 \end{pmatrix} \sim\sim \begin{pmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We get the zero row. I.e., the zero vector can be written as a nontrivial linear combination, hence B is linearly dependent.

b) By finding a nontrivial solution of $a_1\boldsymbol{b}_1 + \cdots + a_n\boldsymbol{b}_n = \mathbf{0}$.

The equation corresponds to a homogeneous system with matrix:

$$\begin{pmatrix} 2 & 4 & 0 & 3 & 0 \\ 1 & 3 & 2 & 4 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 3 & 4 & 1 & 0 & 2 \end{pmatrix} \sim\sim \begin{pmatrix} 2 & 4 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The resulting matrix *contains at least one free variable*: a_3.
A nontrivial solution of the system, e.g. $(4,3,1,0,0)^\mathsf{T}$, yields $4(2,1,0,3)^\mathsf{T} + 3(4,3,1,4)^\mathsf{T} + (0,2,2,1)^\mathsf{T} = \mathbf{0}$, thus B is dependent.

Properties of linear independence

Observation: If B is independent, $C \subseteq B$ then C is independent.

Observation: If C is dependent, $C \subseteq B$ then B is dependent.

Observation: B is independent iff $\forall \boldsymbol{b} \in B : \boldsymbol{v} \notin \operatorname{span}(B \setminus \boldsymbol{v})$.

Proof: $\boldsymbol{b} \in \operatorname{span}(B \setminus \boldsymbol{b}) \;\Leftrightarrow\; \boldsymbol{b} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i$, where $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n \in B \setminus \boldsymbol{b}$.

Proposition: If C is finite generating set of a space V and B is linearly independent in V, then $|B| \leq |C|$.

Proof: Let $C = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_n\}$ and assume for a contrary that there are distinct $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{n+1} \in B$. Express each $\boldsymbol{b}_i$ as $\boldsymbol{b}_i = \sum_{j=1}^{n} a_{ij} \boldsymbol{c}_j$.

The corresponding matrix $\boldsymbol{A}$ has $n+1$ rows and n columns, hence some row is a linear combination of the others.
This combination yields also linear dependence of $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{n+1}$.
Formally: $\exists \boldsymbol{d} = (d_1, \dots, d_{n+1})^\mathsf{T} \in F^{n+1} \setminus \mathbf{0}$: $\boldsymbol{d}^\mathsf{T} \boldsymbol{A} = \mathbf{0}^\mathsf{T} \Rightarrow$

$$\sum_{i=1}^{n+1} d_i \boldsymbol{b}_i = \sum_{i=1}^{n+1} d_i \sum_{j=1}^{n} a_{ij} \boldsymbol{c}_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n+1} d_i a_{ij} \right) \boldsymbol{c}_j = \sum_{j=1}^{n} 0 \boldsymbol{c}_j = \mathbf{0}$$

Distinct ways to describe a vector space

Let $V = \{(0,0,0,0)^\mathsf{T}, (0,1,2,1)^\mathsf{T}, (0,2,1,2)^\mathsf{T}, (1,0,1,0)^\mathsf{T}, (1,1,0,1)^\mathsf{T}, (1,2,2,2)^\mathsf{T}, (2,0,2,0)^\mathsf{T}, (2,1,1,1)^\mathsf{T}, (2,2,0,2)^\mathsf{T}, \}$ be a space of arithmetic vectors over $\mathbb{Z}_3$.

(These vectors viewed as 4-letter words over a 3-letter alphabet have the property that any two words differ in at least two symbols.

Similar sets could be used to design error-correcting codes.)

Could V be described more efficiently than by the list of 9 values?

We may observe that these vectors are dependent, e.g. $(0,0,0,0)^\mathsf{T}$, $(2,1,1,1)^\mathsf{T} = (2,0,2,0)^\mathsf{T} + (0,2,1,2)^\mathsf{T}$ or $(2,0,2,0)^\mathsf{T} = 2 \cdot (1,0,1,0)^\mathsf{T}$.

Repetitive removal of dependent vectors leads to a subset which is independent but still generates the entire V.

Namely, V could be generated just by two vectors, e.g. $(0,1,2,1)^\mathsf{T}$, and $(1,0,1,0)^\mathsf{T}$.

0000	0121	0212
1010	1101	1222
2020	2111	2202

Also, each vector of V is a *unique* linear combination of these two!

Basis

Definition: A *basis* of a vector space V is
a linearly independent set B that generates V.

Why is the concept of a basis so important?

- $\text{span}(B) = V$ imply that every vector of V is a linear combination of vectors of the basis B
- B is linearly independent, hence the above linear combination is *unique* for each vector of V.

Proof: If B is linearly independent and $\boldsymbol{v} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i = \sum_{i=1}^{n} a'_i \boldsymbol{b}_i$, then

$$\mathbf{0} = \boldsymbol{v} - \boldsymbol{v} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i - \sum_{i=1}^{n} a'_i \boldsymbol{b}_i = \sum_{i=1}^{n} (a_i - a'_i) \boldsymbol{b}_i \;\Rightarrow\; \forall i : a_i = a'_i.$$

Definition: Let $B = (\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n)$ be an *ordered* basis of a vector space V over F. The *coordinate vector* of $\boldsymbol{v} \in V$ with respect to the basis B is $[\boldsymbol{v}]_B = (a_1, \ldots, a_n)^{\mathsf{T}} \in F^n$, where $\boldsymbol{v} = \sum_{i=1}^{n} a_i \boldsymbol{b}_i$.

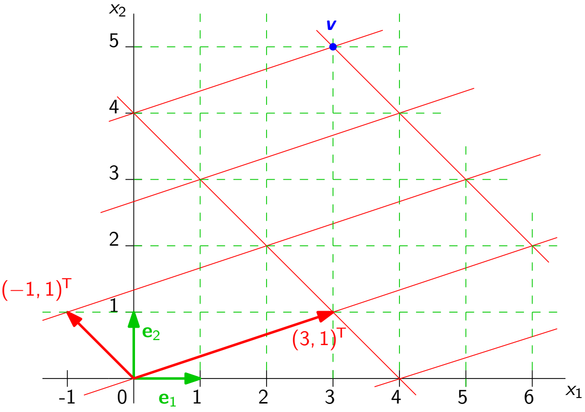
Examples

- In the arithmetic vector space F^n the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of $\mathbf{I}_n$ form the so called *standard basis* E (aka *canonical* or *natural*).
- In $\mathbb{R}^2$, a set $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ is a basis if and only if $\boldsymbol{b}_1 \neq \boldsymbol{b}_2$ and the line determined by $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ does not contain the origin.
- In the vector space of real polynomials, the infinite set $\{x^0, x^1, x^2, \ldots\}$ is an example of infinite basis.
- In the space of polynomials of degree at most 4 we have e.g.: $[x^3 + 2x - 1]_{(x^0, x^1, \ldots, x^4)} = (-1, 2, 0, 1, 0)^\mathsf{T}$, but also $[x^3 + 2x - 1]_{(x^0+x^1, x^1-2x^2, x^2, x^3, x^4)} = (-1, 3, 6, 1, 0)^\mathsf{T}$, as $x^3 + 2x - 1 = -1(x^0 + x^1) + 3(x^1 - 2x^2) + 6x^2 + 1x^3$
- In the vector space $V = \mathcal{P}(S)$ over $\mathbb{Z}_2$ we have e.g. a basis from the single-element sets: $[\{a, c\}]_{(\{a\},\{b\},\{c\})} = (1, 0, 1)^\mathsf{T}$.

Coordinates of a vector with respect to different bases

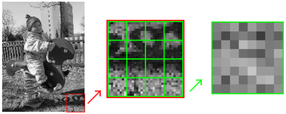
The coordinates of u with respect to the standard (ordered) basis $E = \{\mathbf{e}_1, \mathbf{e}_2\} = \{(1,0)^\mathsf{T}, (0,1)^\mathsf{T}\}$ are: $\boldsymbol{v} = [\boldsymbol{v}]_K = (3,5)^\mathsf{T}$.

With respect to another basis $B = \{(3,1)^\mathsf{T}, (-1,1)^\mathsf{T}\}$ *the same* vector has the coordinates: $[\boldsymbol{v}]_B = (2,3)^\mathsf{T}$.

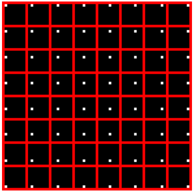


Distinct bases — Jpeg

A vector $\boldsymbol{v}$ is an 8×8 cut from a single color plane and is normalized to $(-128, 127)$:

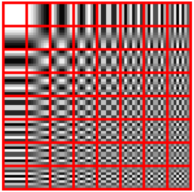


Standard basis E:



$$[\boldsymbol{v}]_{\mathrm{E}} = \begin{pmatrix} 0 & 7 & 30 & -35 & 29 & 1 & -10 & 20 \\ -6 & -54 & -15 & 0 & -18 & -69 & -10 & -32 \\ -38 & 18 & -36 & 58 & 37 & 18 & -7 & -4 \\ 17 & 38 & 27 & -19 & -26 & -43 & -2 & 44 \\ 26 & 33 & 44 & 48 & 42 & 7 & -8 & 20 \\ 11 & 30 & -2 & 32 & 70 & 25 & 25 & 17 \\ 22 & -44 & 30 & -19 & 14 & 48 & 55 & 6 \\ -11 & -16 & 8 & 6 & 22 & -28 & -10 & 17 \end{pmatrix}$$

Basis B from harmonic funtions:



$$[\boldsymbol{v}]_{B} \doteq \begin{pmatrix} 11 & -59 & 16 & 4 & -14 & 4 & -9 & -10 \\ -6 & 110 & -8 & -30 & -30 & 46 & -19 & -13 \\ 16 & -84 & 23 & 5 & -20 & 6 & -12 & -14 \\ -20 & -83 & -29 & -2 & 18 & -4 & 16 & 27 \\ 2 & 91 & 3 & -1 & 29 & -14 & -13 & 21 \\ 27 & 21 & 38 & 41 & -52 & -2 & -40 & 22 \\ -20 & 40 & -28 & 41 & 16 & -46 & -12 & 27 \\ -9 & 59 & -13 & -46 & 11 & 15 & -58 & -39 \end{pmatrix}$$

These are 64-dimensional vectors, only depicted as matrices.

Lossy compression and decompression (simplified)

$$\begin{pmatrix} 11 & -59 & 16 & 4 & -14 & 4 & -9 & -10 \\ -6 & 110 & -8 & -30 & -30 & 46 & -19 & -13 \\ 16 & -84 & 23 & 5 & -20 & 6 & -12 & -14 \\ -20 & -83 & -29 & -2 & 18 & -4 & 16 & 27 \\ 2 & 91 & 3 & -1 & 29 & -14 & -13 & 21 \\ 27 & 21 & 38 & 41 & -52 & -2 & -40 & 22 \\ -20 & 40 & -28 & 41 & 16 & -46 & -12 & 27 \\ -9 & 59 & -13 & -46 & 11 & 15 & -58 & -39 \end{pmatrix}$$

divide component-wise by the so-called *quantization matrix*

$$\begin{pmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{pmatrix}$$

and round

$$\begin{pmatrix} 3 & -1 & -2 & -1 & 1 & 0 & 0 & 1 \\ -5 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -4 & -2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

multiply by the quant. matrix

$$\begin{pmatrix} 48 & -11 & -20 & -16 & 24 & 0 & 0 & 61 \\ -60 & 24 & 0 & -19 & 0 & 0 & 0 & 0 \\ -56 & -26 & 0 & 24 & 0 & 0 & 0 & 56 \\ 70 & 0 & 0 & 29 & 0 & 0 & 0 & 0 \\ 18 & 44 & 37 & -56 & 0 & 0 & 0 & 0 \\ 24 & -35 & 55 & 0 & -81 & 0 & 0 & 0 \\ 49 & 0 & -78 & 0 & 0 & 0 & 0 & 0 \\ 72 & 0 & -95 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

convert from the basis B to E

$$\begin{pmatrix} 1 & 0 & 39 & -16 & 23 & -13 & 23 & 0 \\ 1 & -53 & -40 & -38 & -3 & -61 & -36 & -14 \\ -41 & -15 & 32 & 47 & 55 & 20 & -2 & -24 \\ 15 & 33 & 12 & -38 & -71 & -36 & 12 & 43 \\ 31 & 27 & 48 & 82 & 56 & 5 & -11 & 18 \\ 22 & -1 & 6 & 23 & 53 & 35 & 25 & 13 \\ 1 & -5 & 7 & -33 & 25 & 38 & 62 & 21 \\ -2 & -29 & 17 & -3 & 34 & -38 & -4 & 5 \end{pmatrix}$$

Data: 27 ints. $\in \{-5, \ldots, 5\} \setminus 0$

Average hue deviation $< 6\%$

Original:  Restored: 

Existence of a basis

Observation: A set B is a basis of a vector space V,
if and only if $\text{span}(B) = V$ and $\forall \boldsymbol{b} \in B : \boldsymbol{b} \notin \text{span}(B \setminus \boldsymbol{b})$.

Corollary: Every finite generating set C of a vector space V contains a basis B as a subset.

Proof: First set $B = C$. Then iteratively test all $\boldsymbol{b} \in B$ whether $\boldsymbol{b} \in \text{span}(B \setminus \boldsymbol{b})$. If so then remove $\boldsymbol{b}$ from B.

Theorem: Every vector space has a basis.

... for finitely generated it is proven above;
for infinitely generated we omit a proof.
(This part of the theorem is equivalent to the axiom of choice.)

Questions to understand the lecture topic

- How can we select as many linearly independent columns as possible from a matrix in echelon form?
- How would you test whether a set of even subgraphs is linearly independent?
- Does the uniqueness of the coefficients of the linear combination hold for the linearly independent set B, even if B is infinite and the combinations are expressed relative to other subsets of B?
- What does a basis of the space of matrices $F^{m \times n}$ look like?