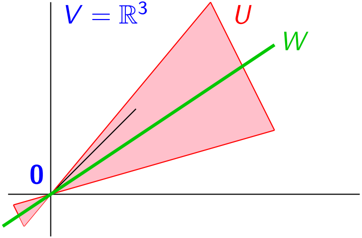
Subspace

Definition: Let V be a vector space over F then
a *subspace* U is a nonempty subset of V satisfying:

- $\forall \boldsymbol{u}, \boldsymbol{v} \in U : \boldsymbol{u} + \boldsymbol{v} \in U$
- $\forall \boldsymbol{v} \in U \ \forall a \in F : a\boldsymbol{v} \in U$

Example: A plane U through the origin $\mathbf{0}$ is a subspace of $V = \mathbb{R}^3$.

A line $W \subset U$ through the origin is a subspace of U and of V as well.



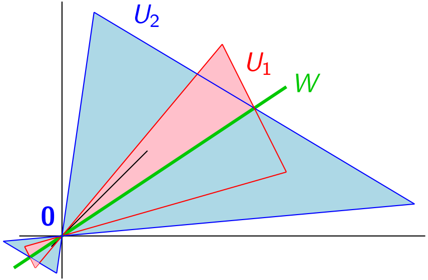
Polynomials of degree ≤ 5 form a subspace of the function space.

Observation: Any subspace is also a vector space, because the existential axioms follow the closure: $\mathbf{0} = 0\boldsymbol{v} \in U$ and $-\boldsymbol{v} = (-1)\boldsymbol{v} \in U$. Other axioms hold already on V.

Intersection of subspaces

Theorem: Let $(U_i, i \in I)$ be any system of subspaces of a space V. The intersection of this system $\bigcap_{i \in I} U_i$ is also a subspace of V.

Example: Planes U_1 and U_2 are subspaces of the vector space $\mathbb{R}^3$. Their intersection is the line W. It is also a subspace of $\mathbb{R}^3$.



Intersection of subspaces

Theorem: Let $(U_i, i \in I)$ be any system of subspaces of a space V. The intersection of this system $\bigcap_{i \in I} U_i$ is also a subspace of V.

Proof: Let $W = \bigcap_{i \in I} U_i$. We show that W is closed on $+$ and $\cdot$.

$\forall \boldsymbol{u}, \boldsymbol{v} \in W :$
$\boldsymbol{u}, \boldsymbol{v} \in W \Rightarrow \forall i \in I : \boldsymbol{u}, \boldsymbol{v} \in U_i \Rightarrow \forall i \in I : \boldsymbol{u} + \boldsymbol{v} \in U_i \Rightarrow \boldsymbol{u} + \boldsymbol{v} \in W$

$\forall a \in F, \boldsymbol{v} \in W :$
$\boldsymbol{v} \in W \Rightarrow \forall i \in I : \boldsymbol{v} \in U_i \Rightarrow \forall i \in I : a\boldsymbol{v} \in U_i \Rightarrow a\boldsymbol{v} \in W$

Note that when $I = \emptyset$, then the empty intersection $W = \bigcap_{i \in \emptyset} U_i = V$ is a subspace of V itself.

Subspace generated by a set, linear combination

Definition: The subspace of a vector space V *generated* by a set S is the intersection of all subspaces U of V that contain S.

Formally $\mathrm{span}(S) = \bigcap\{U : S \subseteq U, U \text{ is a subspace of } V\}$

It is also called *linear hull/span* of S and may be denoted by $\mathcal{L}(S)$.

Examples: For $V = \mathbb{R}^3$, $\mathrm{span}(\{(2,2,2)^\mathsf{T}\}) = \{(a,a,a)^\mathsf{T}, a \in \mathbb{R}\}$
... the line containing points having all three coordinates identical

$\mathrm{span}(\{(1,0,0)^\mathsf{T}, (0,1,0)^\mathsf{T}\}) = \{(a,b,0)^\mathsf{T}, a, b \in \mathbb{R}\}$
... the plane determined by the first two axes

Definition: A *linear combination* of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in V$ over F is any vector $\boldsymbol{u} = a_1\boldsymbol{v}_1 + \ldots + a_k\boldsymbol{v}_k$ where $a_1, \ldots, a_k \in F$.

Theorem: If V is a vector space over F and S is a subset of V, then $\mathrm{span}(S)$ is the set of all linear combinations of vectors from S.

Example for $W = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$

For $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 4 & 0 & 3 \\ -2 & 4 & -6 & 6 & 0 \\ 4 & -8 & 13 & -9 & -4 \\ 3 & -6 & 9 & -9 & 1 \end{pmatrix} \sim\sim \begin{pmatrix} 1 & -2 & 4 & 0 & 3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

we solve $W = \{p_1(2,1,0,0,0)^\mathsf{T} + p_2(12,0,-3,1,0)^\mathsf{T} : p_1, p_2 \in \mathbb{R}\}$

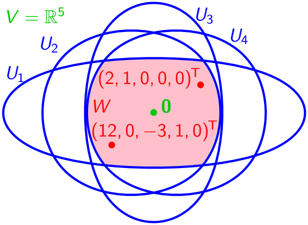
1. W could be viewed the *intersection* of the following subspaces:

$U_1 = \{\boldsymbol{x} : (1,-2,4,0,3) \cdot \boldsymbol{x} = 0\}$
$U_2 = \{\boldsymbol{x} : (-2,4,-6,6,0) \cdot \boldsymbol{x} = 0\}$
$U_3 = \{\boldsymbol{x} : (4,-8,13,-9,-4) \cdot \boldsymbol{x} = 0\}$
$U_4 = \{\boldsymbol{x} : (3,-6,9,-9,1) \cdot \boldsymbol{x} = 0\}$

Each U_i corresponds to one equation of the system and is a *hyperplane* in $\mathbb{R}^5$.
$W = U_1 \cap U_2 \cap U_3 \cap U_4$

2. The same subspace W was already described as a *set of linear combinations*
$W = \operatorname{span}((2,1,0,0,0)^\mathsf{T}, (12,0,-3,1,0)^\mathsf{T})$

Correctness of Gaussian elimination and backward substitution yields that both ways describe the same space in $\mathbb{R}^5$.



Proof of the theorem

Theorem: If V is a vector space over F and S is a subset of V, then span(S) is the set of all linear combinations of vectors from S.

Denote $W_1 = \bigcap\{U : S \subseteq U, U \text{ is subspace of } V\} = \text{span}(S)$ and

$W_2 = \left\{ \sum_{i=1}^{k} a_i \boldsymbol{v}_i : k \in \mathbb{N}, a_i \in F, \boldsymbol{v}_i \in S \right\}$. Goal $W_1 = W_2$.

W_2 is a subspace, since it is closed on multiples by a scalar $b \in F$:

$$\boldsymbol{u} \in W_2 \Rightarrow \boldsymbol{u} = \sum_{i=1}^{k} a_i \boldsymbol{v}_i \Rightarrow b\boldsymbol{u} = b \sum_{i=1}^{k} a_i \boldsymbol{v}_i = \sum_{i=1}^{k} (ba_i) \boldsymbol{v}_i \Rightarrow b\boldsymbol{u} \in W_2$$

analogously also on addition: $\boldsymbol{u}, \boldsymbol{u}' \in W_2 \Rightarrow \ldots \Rightarrow \boldsymbol{u} + \boldsymbol{u}' \in W_2$

Since $S \subseteq W_2$, we have W_2 among the intersecting subspaces U. Hence $W_1 \subseteq W_2$.

Every U contains S and is closed on addition and scalar multiples. Thus every U contains all linear combinations of vectors of S. Hence $\forall U : W_2 \subseteq U \Rightarrow W_2 \subseteq W_1$.

Questions to understand the lecture topic

- Can an infinite number of subspaces have a non-empty intersection?
- If $\boldsymbol{u}$ is a linear combination of vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ and $\boldsymbol{u}'$ is a linear combination of $\boldsymbol{w}_1, \dots, \boldsymbol{w}_l$, can the sum $\boldsymbol{u} + \boldsymbol{u}'$ be expressed as a linear combination of vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{w}_1, \dots, \boldsymbol{w}_l$?

  How does this relate to the linear hull of an infinite set S?