Vector space

Definition: A *vector space* $(V, +, \cdot)$ over a field $(F, +, \cdot)$ is a set V with a binary operation $+$ on V and a binary operation $\cdot : F \times V \to V$ called *scalar multiplication* s.t.:

- $(V, +)$ is an Abelian group
- $\forall a, b \in F \ \forall \boldsymbol{v} \in V : (a \cdot b) \cdot \boldsymbol{v} = a \cdot (b \cdot \boldsymbol{v})$
- $\forall \boldsymbol{v} \in V : 1 \cdot \boldsymbol{v} = \boldsymbol{v}$
- $\forall a, b \in F \ \forall \boldsymbol{v} \in V : (a + b) \cdot \boldsymbol{v} = (a \cdot \boldsymbol{v}) + (b \cdot \boldsymbol{v})$
- $\forall a \in F \ \forall \boldsymbol{u}, \boldsymbol{v} \in V : a \cdot (\boldsymbol{u} + \boldsymbol{v}) = (a \cdot \boldsymbol{u}) + (a \cdot \boldsymbol{v})$

Elements of F are called *scalars*, elements of V are called *vectors*.

Distinguish the zero scalar $0 \in F$ and the zero vector $\mathbf{0} \in V$.
We have opposite scalars $-a \in F$ and opposite vectors $-\boldsymbol{v} \in V$.

There are inverse scalars $a^{-1} \in F$ but *no inverse vectors* $\boldsymbol{v}^{-1}$*!*
Products $\boldsymbol{v} \cdot a$ and $\boldsymbol{v} \cdot \boldsymbol{u}$ are not defined and are *formally wrong!*

The product symbol $\cdot$ is often omitted and has priority to $+$.

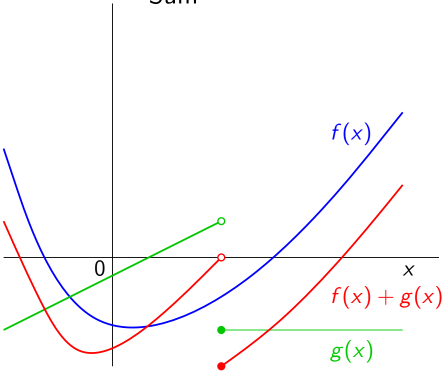
Examples

- The *arithmetic* vector space F^n of dimension n over F.
  Vectors are ordered n-tuples of elements of F.
  Additions and scalar multiplications are done componentwise.
  Any field F yields the vector space F^1 of the same cardinality.
- $F^{m \times n}$ ... matrices of order $m \times n$ over F.
- $V = \{\mathbf{0}\}$ the *trivial* vector space over any field F.
- Polynomials with coefficients in F.
- Polynomials of bounded degree.

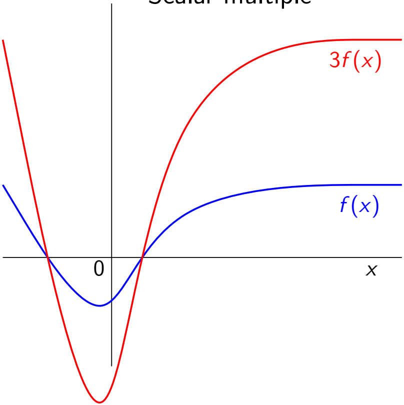
The vector space of real functions over the field $\mathbb{R}$

Vectors f, g are the real functions of real variable

Sum



Scalar multiple



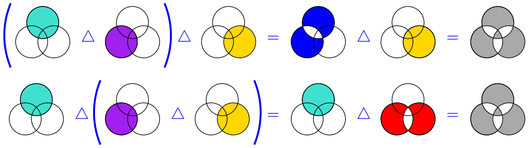
Set systems as vector spaces

Let $\mathcal{A}$ be a system of subsets of a set S that is closed under taking symmetric difference $\triangle$.

Then $(\mathcal{A}, \triangle, \cdot)$ is a vector space over $\mathbb{Z}_2$, where $\cdot$ is defined as: $0 \cdot A = \emptyset$, the neutral element of $\triangle$, and $1 \cdot A = A$ for all $A \in \mathcal{A}$.

Observe that $\forall A, B, C \in \mathcal{A} : (A \triangle B) \triangle C = A \triangle (B \triangle C)$,

since the result contains those elements of S that belong to an odd number of sets A, B and C.



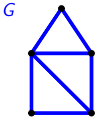
We may represent any $A \in \mathcal{A}$ by its *characteristic function*

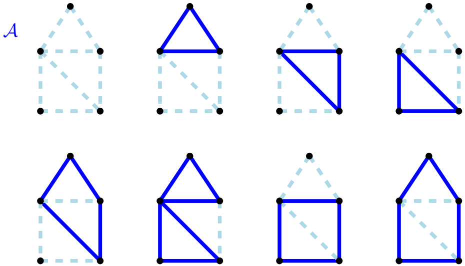
$\chi_A : S \to \mathbb{Z}_2$ defined as $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

Observation: The set $A \triangle B$ has the characteristic function $\chi_{A \triangle B} = \chi_A + \chi_B$, since $1 + 1 = 0$ in $\mathbb{Z}_2$.

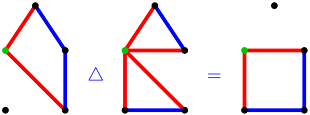
A vector space on graphs

Let $S = E_G$ and $\mathcal{A}$ contain those edge sets A such that every vertex of G is incident with an even number of edges in A. Such sets A induce the so called *even subgraphs* of G.





Observe that $\triangle$ preserves the even degree, since the symmetric difference of two sets of even cardinality, namely the edges incident to a vertex, has also an even cardinality.



$$|A \triangle B| = |A| + |B| - 2|A \cap B|$$

Properties of vector spaces

Since $(V, +)$ is a group, we have already proved:

- uniqueness of the vector $\mathbf{0}$,
- uniqueness of $-\boldsymbol{u}$,
- correctness of equiv. transforms: $\boldsymbol{u} = \boldsymbol{v} \Leftrightarrow \boldsymbol{u} + \boldsymbol{w} = \boldsymbol{v} + \boldsymbol{w}$,
- solubility of equations: $\boldsymbol{x} + \boldsymbol{u} = \boldsymbol{v} \Leftrightarrow \boldsymbol{x} = \boldsymbol{v} - \boldsymbol{u}$.

Observation: For any $\boldsymbol{v} \in V$ and $a \in F$ holds: $0\boldsymbol{v} = a\mathbf{0} = \mathbf{0}$.

Proof:

$0\boldsymbol{v} = 0\boldsymbol{v} + \mathbf{0} = 0\boldsymbol{v} + 0\boldsymbol{v} - 0\boldsymbol{v} = (0 + 0)\boldsymbol{v} - 0\boldsymbol{v} = 0\boldsymbol{v} - 0\boldsymbol{v} = \mathbf{0}$

$a\mathbf{0} = a\mathbf{0} + \mathbf{0} = a\mathbf{0} + a\mathbf{0} - a\mathbf{0} = a(\mathbf{0} + \mathbf{0}) - a\mathbf{0} = a\mathbf{0} - a\mathbf{0} = \mathbf{0}$

Observation: For any $\boldsymbol{v} \in V$ holds $(-1)\boldsymbol{v} = -\boldsymbol{v}$.

Proof: $(-1)\boldsymbol{v} + \boldsymbol{v} = (-1)\boldsymbol{v} + 1\boldsymbol{v} = (-1 + 1)\boldsymbol{v} = 0\boldsymbol{v} = \mathbf{0}$

Observation: If $a\boldsymbol{v} = \mathbf{0}$ then $a = 0$ or $\boldsymbol{v} = \mathbf{0}$.

Proof: If $a \neq 0$ then $\boldsymbol{v} = 1\boldsymbol{v} = a^{-1}a\boldsymbol{v} = a^{-1}\mathbf{0} = \mathbf{0}$.

Questions to understand the lecture topic

- Which axioms can be used when working in vector spaces, including those in the definition of a field and a group?
- Which operations can be performed with arithmetic vectors?
- Which operations with polynomials are vector space operations?
- How would you express the solution to the equation $a\boldsymbol{x} + \boldsymbol{u} = \boldsymbol{v}$?
  Which axioms yield the formula for the solution?