Regular, singular and inverse matrices

Definition: If for $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ there exists $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ such that $\boldsymbol{AB} = \mathbf{I}_n$, then $\boldsymbol{B}$ is called the *inverse matrix* and denoted by $\boldsymbol{A}^{-1}$. If $\boldsymbol{A}$ has an inverse, then it is called *regular*, otherwise it is *singular*.

Theorem: For $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ the following are equivalent:

1. $\boldsymbol{A}$ is regular, i.e. $\exists \boldsymbol{B} \in \mathbb{R}^{n \times n} : \boldsymbol{AB} = \mathbf{I}_n$.
2. $\text{rank}(\boldsymbol{A}) = n$.
3. $\boldsymbol{A} \sim\sim \mathbf{I}_n$.
4. The system $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution $\boldsymbol{x} = \mathbf{0}$.

Proof: 2. $\Leftrightarrow$ 4. was already proved.
2. $\Rightarrow$ 3. by Gauss-Jordan elimination, 2. $\Leftarrow$ 3. $\mathbf{I}_n$ is in REF.
2. $\Rightarrow$ 1. Denote $\mathbf{I}_n = (\mathbf{e}_1 | \dots | \mathbf{e}_n)$. For $i = 1, \dots, n$ consider systems $\boldsymbol{Ax}_i = \mathbf{e}_i$. Since $\text{rank}(\boldsymbol{A}) = n$ we get $\boldsymbol{B} = (\boldsymbol{x}_1 | \dots | \boldsymbol{x}_n)$.
1. $\Rightarrow$ 2. If $\text{rank}(\boldsymbol{A}) < n$ then for some i, the i-th row of $\boldsymbol{A}$ can be eliminated by the other rows, thus $\boldsymbol{Ax}_i = \mathbf{e}_i$ has no solution as the only 1 in $\mathbf{e}_i$ cannot be eliminated by zeroes.

Example

$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 4 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \mathrm{III} \\ \mathrm{II}-3\mathrm{I} \end{matrix} \sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 5 & 6 \end{pmatrix} \begin{matrix} \\ \\ -5\mathrm{II} \end{matrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{matrix} +\mathrm{III} \\ -\mathrm{III} \\ \\ \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The third matrix in REF yields $\operatorname{rank}(\boldsymbol{A}) = 3$, hence $\boldsymbol{A}$ is regular.

The system $\boldsymbol{A}\boldsymbol{x}_1 = \mathbf{e}_1$ has a solution

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 3 & 4 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right) \qquad \boldsymbol{x}_1 = \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}$$

Analogously systems $\boldsymbol{A}\boldsymbol{x}_2 = \mathbf{e}_2$ and $\boldsymbol{A}\boldsymbol{x}_3 = \mathbf{e}_3$ have solutions $\boldsymbol{x}_2 = (-1, 1, -1)^\mathsf{T}$ and $\boldsymbol{x}_3 = (-8, 6, -5)^\mathsf{T}$.

We arrange them into the inverse matrix: $\boldsymbol{A}^{-1} = \begin{pmatrix} 4 & -1 & -8 \\ -3 & 1 & 6 \\ 3 & -1 & -5 \end{pmatrix}$

$$\boldsymbol{A}' = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} \mathrm{II} \\ \mathrm{III} \\ \mathrm{I}-\frac{1}{3}\mathrm{II}-2\mathrm{III} \end{matrix} \sim \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \operatorname{rank}(\boldsymbol{A}') = 2 \;\Rightarrow$$

$\boldsymbol{A}'$ is singular.

The system $\boldsymbol{A}'\boldsymbol{x}_1 = \mathbf{e}_1$ has no solution $\Rightarrow$ $(\boldsymbol{A}')^{-1}$ does not exist.

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 3 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \begin{matrix} \mathrm{II} \\ \mathrm{III} \\ \mathrm{I}-\frac{1}{3}\mathrm{II}-2\mathrm{III} \end{matrix} \sim \left(\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Properties of inverse matrix

Corollary: If the inverse matrix $\boldsymbol{A}^{-1}$ exists, it is unique.

Theorem: The inverse matrix satisfies: $\boldsymbol{A}^{-1}\boldsymbol{A} = \mathbf{I}$.

Proof: We first show that $\boldsymbol{A}^{-1}$ is regular:
If $\boldsymbol{A}^{-1}\boldsymbol{x} = \mathbf{0}$ has a solution then $\boldsymbol{x} = \mathbf{I}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{A}^{-1}\boldsymbol{x} = \boldsymbol{A}\mathbf{0} = \mathbf{0}$.
Hence there exists $(\boldsymbol{A}^{-1})^{-1}$ and we get:
$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}^{-1}\boldsymbol{A}\mathbf{I} = (\boldsymbol{A}^{-1}\boldsymbol{A})(\boldsymbol{A}^{-1}(\boldsymbol{A}^{-1})^{-1}) =$
$\boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{A}^{-1})(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}^{-1}\mathbf{I}(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1})^{-1} = \mathbf{I}$

Corollary: If $\boldsymbol{B}\boldsymbol{A} = \mathbf{I}$ then $\boldsymbol{A}^{-1} = \boldsymbol{B}$.

Proof: $\boldsymbol{B}\boldsymbol{A} = \mathbf{I} \implies \boldsymbol{A}\boldsymbol{B} = \mathbf{I} \implies \boldsymbol{A}^{-1} = \boldsymbol{B}$

Inverse matrix calculation

- Assemble $(\boldsymbol{A}|\mathbf{I})$ and by Gauss–Jordan elimination get $(\mathbf{I}|\boldsymbol{B})$.
- If this process fails, then $\boldsymbol{A}$ is singular.
- Denote $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_k$ the elementary matrices of the applied elementary transforms. Then the left side of $(\boldsymbol{A}|\mathbf{I}) \sim\sim (\mathbf{I}|\boldsymbol{B})$ yields $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \mathbf{I}$, the right side yields $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \mathbf{I} = \boldsymbol{B}$ thus $\boldsymbol{B}\boldsymbol{A} = \mathbf{I}$ and therefore $\boldsymbol{A}^{-1} = \boldsymbol{B}$.
- The columns of $\boldsymbol{B}$ are in fact solutions of systems $\boldsymbol{A}\boldsymbol{x}_i = \mathbf{e}_i$.

Example:

$$(\boldsymbol{A}|\mathbf{I}) = \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \underset{3\text{I}-\text{II}}{\overset{\text{III}}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 5 & 6 & 3 & -1 & 0 \end{array}\right) \underset{-5\text{II}}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & -1 & -5 \end{array}\right) \begin{array}{c} +\text{III} \\ -\text{III} \end{array}\!\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -8 \\ 0 & 1 & 0 & -3 & 1 & 6 \\ 0 & 0 & 1 & 3 & -1 & -5 \end{array}\right) = (\mathbf{I}|\boldsymbol{A}^{-1})$$

Verification: $$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 4 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 & -8 \\ -3 & 1 & 6 \\ 3 & -1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Properties of regular matrices

Observation: If $\boldsymbol{R}$ is regular, then:

$$\boldsymbol{A} = \boldsymbol{B} \iff \boldsymbol{AR} = \boldsymbol{BR} \iff \boldsymbol{RA} = \boldsymbol{RB} \quad \not\Leftrightarrow \quad \boldsymbol{AR} = \boldsymbol{RB}$$

Proof: $\Rightarrow$ trivially, $\Leftarrow$: $\boldsymbol{A} = \boldsymbol{A}\mathbf{I} = \boldsymbol{ARR}^{-1} = \boldsymbol{BRR}^{-1} = \boldsymbol{B}\mathbf{I} = \boldsymbol{B}$.

The other equivalence in the same way.

Proposition: Regular matrices $\boldsymbol{A}, \boldsymbol{B}$ of the same order satisfy:

- $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
- $\boldsymbol{AB}$ is regular
- $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- $(\boldsymbol{A}^{\mathsf{T}})^{-1} = (\boldsymbol{A}^{-1})^{\mathsf{T}}$

Proof: $(\boldsymbol{A}^{-1})^{-1} = (\boldsymbol{A}^{-1})^{-1}\mathbf{I} = (\boldsymbol{A}^{-1})^{-1}\boldsymbol{A}^{-1}\boldsymbol{A} = \mathbf{I}\boldsymbol{A} = \boldsymbol{A}$.

Prove the remaining claims on your own by analogous arguments.

Corollary: For regular $\boldsymbol{A}, \boldsymbol{B}$ it holds: $\boldsymbol{A} = \boldsymbol{B} \iff \boldsymbol{A}^{-1} = \boldsymbol{B}^{-1}$.
(In other words, inverting regular matrices is an equivalent transformation on equations.)

Matrix equations

Observation: For matrices of identical or compatible types:

$$\begin{array}{lll}
& \boldsymbol{A}+\boldsymbol{X}=\boldsymbol{B} & \Leftrightarrow \quad \boldsymbol{X}=\boldsymbol{B}-\boldsymbol{A}=\boldsymbol{B}+(-1)\boldsymbol{A} \\
\text{for } t\neq 0: & t\boldsymbol{X}=\boldsymbol{B} & \Leftrightarrow \quad \boldsymbol{X}=\frac{1}{t}\boldsymbol{B} \\
\text{for regular } \boldsymbol{A}: & \boldsymbol{A}\boldsymbol{X}=\boldsymbol{B} & \Leftrightarrow \quad \boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B} \\
\text{for regular } \boldsymbol{A}: & \boldsymbol{X}\boldsymbol{A}=\boldsymbol{B} & \Leftrightarrow \quad \boldsymbol{X}=\boldsymbol{B}\boldsymbol{A}^{-1}
\end{array}$$

Beware, products $\boldsymbol{A}^{-1}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}^{-1}$ could be distinct.

Example:

Equations $\begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix}\boldsymbol{X}=\begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$ and $\boldsymbol{Y}\begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix}=\begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$

have different solutions $\boldsymbol{X}=\begin{pmatrix} -9 & 3 \\ 42 & -14 \end{pmatrix}$ and $\boldsymbol{Y}=\begin{pmatrix} 7 & -15 \\ 14 & -30 \end{pmatrix}$

Test:

$$\begin{array}{cc|cc}
 & & -9 & 3 \\
 & & 42 & -14 \\
\hline
9 & 2 & 3 & -1 \\
4 & 1 & 6 & -2
\end{array}
\qquad\qquad
\begin{array}{cc|cc}
 & & 9 & 2 \\
 & & 4 & 1 \\
\hline
7 & -15 & 3 & -1 \\
14 & -30 & 6 & -2
\end{array}$$

Questions to understand the lecture topic

- Which matrix operations preserve the properties "being regular"and "being singular"?