Operations with matrices

Definitions: For any $m, n \in \mathbb{N}$ we define the *zero matrix* $\mathbf{0} \in \mathbb{R}^{m\times n}$ satisfying $\forall i, j : (\mathbf{0})_{i,j} = 0$. For $n \in \mathbb{N}$ the *identity matrix* $\boldsymbol{I}_n$ is defined as $(\boldsymbol{I}_n)_{i,j} = 1$ when $i = j$ and $(\boldsymbol{I}_n)_{i,j} = 0$ otherwise.
The *main diagonal* of a square matrix $\boldsymbol{A}$ means the elements $a_{i,i}$.

The *transpose* of a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is $\boldsymbol{A}^T \in \mathbb{R}^{n\times m}$ defined as $(\boldsymbol{A}^T)_{i,j} = a_{j,i}$. A square matrix $\boldsymbol{A}$ is *symmetric* if $\boldsymbol{A}^T = \boldsymbol{A}$.

The *sum* of matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}$ is $(\boldsymbol{A} + \boldsymbol{B}) \in \mathbb{R}^{m\times n}$ defined as

$$(\boldsymbol{A} + \boldsymbol{B})_{i,j} = a_{i,j} + b_{i,j}.$$

The *α-multiple* of a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\alpha \in \mathbb{R}$ is $(\alpha\boldsymbol{A}) \in \mathbb{R}^{m\times n}$ s.t.

$$(\alpha\boldsymbol{A})_{i,j} = \alpha a_{i,j}.$$

For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times p}$ the *product* $(\boldsymbol{AB}) \in \mathbb{R}^{m\times p}$ is defined as

$$(\boldsymbol{AB})_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Example of matrix product

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 0 & 1 & 3 \\ 3 & 1 & 2 & 0 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{AB} = \begin{pmatrix} 9 & 8 \\ 2 & 3 \\ 7 & 9 \end{pmatrix}$$

Mnemotechnically:

$$\begin{array}{c|c} & \boldsymbol{B} \\ \hline \boldsymbol{A} & \boldsymbol{AB} \end{array}$$

$$\begin{array}{cccc|cc} & & & & 1 & 2 \\ & & & & 0 & 3 \\ & & & & 2 & 0 \\ & & & & 0 & 1 \\ \hline 1 & 2 & 4 & 0 & 9 & 8 \\ 0 & 0 & 1 & 3 & 2 & 3 \\ 3 & 1 & 2 & 0 & 7 & 9 \end{array}$$

$$\begin{array}{cccc|cc} & & & & 1 & . \\ & & & & 0 & . \\ & & & & 2 & . \\ & & & & 0 & . \\ \hline . & . & . & . & . & . \\ . & . & . & . & . & . \\ 3 & 1 & 2 & 0 & 7 & . \end{array}$$

$$3 \cdot 1 + 1 \cdot 0 + 2 \cdot 2 + 0 \cdot 0 = 7$$

Observation: The matrix product $\boldsymbol{AB}$ for $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{B} \in \mathbb{R}^{n \times p}$ requires mnp multiplications and $m(n-1)p$ additions $\approx mnp$ arithmetic operations.

Prove or solve on your own

Proposition: If the results of operations are defined then:

$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$ $\quad$ $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$

$\boldsymbol{A}+\mathbf{0}=\boldsymbol{A}$ $\quad$ $\exists!\boldsymbol{B}:\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{A}$

$\alpha(\beta\boldsymbol{A})=(\alpha\beta)\boldsymbol{A}$ $\quad$ $(\alpha+\beta)\boldsymbol{A}=\alpha\boldsymbol{A}+\beta\boldsymbol{A}$

$(\boldsymbol{A}+\boldsymbol{B})^T=\boldsymbol{A}^T+\boldsymbol{B}^T$ $\quad$ $(\alpha\boldsymbol{A})^T=\alpha\boldsymbol{A}^T$

$(\boldsymbol{A}^T)^T=\boldsymbol{A}$ $\quad$ Pf: $((A^T)^T)_{i,j}=(A^T)_{j,i}=a_{i,j}$

Find square matrices $\boldsymbol{A},\boldsymbol{B}$ such that $\boldsymbol{AB}\neq\boldsymbol{BA}$.

Show that the matrix $\boldsymbol{AA}^T$ is symmetric for any $\boldsymbol{A}$.

Show that for any $\boldsymbol{A}\in\mathbb{R}^{m\times n}$ it holds that $\boldsymbol{I}_m\boldsymbol{A}=\boldsymbol{AI}_n=\boldsymbol{A}$.

Derive the following rules for the products of block matrices.

$$\begin{pmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{B}_1\\ \boldsymbol{B}_2\end{pmatrix}=\boldsymbol{A}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{B}_2$$

$$\begin{pmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ \boldsymbol{A}_3 & \boldsymbol{A}_4\end{pmatrix}\begin{pmatrix}\boldsymbol{B}_1 & \boldsymbol{B}_2\\ \boldsymbol{B}_3 & \boldsymbol{B}_4\end{pmatrix}=\begin{pmatrix}\boldsymbol{A}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{B}_3 & \boldsymbol{A}_1\boldsymbol{B}_2+\boldsymbol{A}_2\boldsymbol{B}_4\\ \boldsymbol{A}_3\boldsymbol{B}_1+\boldsymbol{A}_4\boldsymbol{B}_3 & \boldsymbol{A}_3\boldsymbol{B}_2+\boldsymbol{A}_4\boldsymbol{B}_4\end{pmatrix}$$

Proposition: If the results of the operations are defined then:

1. $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$
2. $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
3. $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$
4. $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}$

Proof: 1. $\boldsymbol{A}$ is of order $m \times n \implies \boldsymbol{B}$ is of order $n \times p$

$$((\boldsymbol{AB})^T)_{i,j} = (\boldsymbol{AB})_{j,i} = \sum_{k=1}^{n} a_{j,k} b_{k,i} =$$

$$= \sum_{k=1}^{n} b_{k,i} a_{j,k} = \sum_{k=1}^{n} (\boldsymbol{B}^T)_{i,k} (\boldsymbol{A}^T)_{k,j} = (\boldsymbol{B}^T\boldsymbol{A}^T)_{i,j}$$

2. $\boldsymbol{A}$ of order $m \times n \implies \boldsymbol{B}$ of order $n \times p \implies \boldsymbol{C}$ of order $p \times q$

$$((\boldsymbol{AB})\boldsymbol{C})_{i,j} = \sum_{k=1}^{p} (\boldsymbol{AB})_{i,k} c_{k,j} = \sum_{k=1}^{p} \left(\sum_{l=1}^{n} a_{i,l} b_{l,k} \right) c_{k,j} =$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} a_{i,l} b_{l,k} c_{k,j} = \sum_{l=1}^{n} \sum_{k=1}^{p} a_{i,l} b_{l,k} c_{k,j} =$$

$$= \sum_{l=1}^{n} a_{i,l} \left(\sum_{k=1}^{p} b_{l,k} c_{k,j} \right) = \sum_{l=1}^{n} a_{i,l} (\boldsymbol{BC})_{l,j} = (\boldsymbol{A}(\boldsymbol{BC}))_{i,j}$$

Proposition: If the results of the operations are defined then:

1. $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$
2. $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
3. $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$
4. $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}$

Proof: 3. $\boldsymbol{A} \in \mathbb{R}^{m\times n} \Longrightarrow \boldsymbol{B} \in \mathbb{R}^{m\times n}, \boldsymbol{C} \in \mathbb{R}^{n\times p}$

$$((\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C})_{i,j} = \sum_{k=1}^{n} (\boldsymbol{A}+\boldsymbol{B})_{i,k} c_{k,j} = \sum_{k=1}^{n} (a_{i,k}+b_{i,k}) c_{k,j} =$$

$$= \sum_{k=1}^{n} (a_{i,k}c_{k,j} + b_{i,k}c_{k,j}) = \sum_{k=1}^{n} a_{i,k}c_{k,j} + \sum_{k=1}^{n} b_{i,k}c_{k,j} =$$

$$= (\boldsymbol{AC})_{i,j} + (\boldsymbol{BC})_{i,j} = (\boldsymbol{AC}+\boldsymbol{BC})_{i,j}$$

4. $\boldsymbol{A} \in \mathbb{R}^{m\times n} \Longrightarrow \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{n\times p}$

$$(\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}))_{i,j} = \sum_{k=1}^{n} a_{i,k} (\boldsymbol{B}+\boldsymbol{C})_{k,j} = \sum_{k=1}^{n} a_{i,k}(b_{k,j}+c_{k,j}) =$$

$$= \sum_{k=1}^{n} (a_{i,k}b_{k,j} + a_{i,k}c_{k,j}) = \sum_{k=1}^{n} a_{i,k}b_{k,j} + \sum_{k=1}^{n} a_{i,k}c_{k,j} =$$

$$= (\boldsymbol{AB})_{i,j} + (\boldsymbol{AC})_{i,j} = (\boldsymbol{AB}+\boldsymbol{AC})_{i,j}$$

Example for the proof of the product associativity

$$((\boldsymbol{AB})\boldsymbol{C})_{i,j} = \sum_{k=1}^{p} (\boldsymbol{AB})_{i,k} c_{k,j} = \sum_{k=1}^{p} \left(\sum_{l=1}^{n} a_{i,l} b_{l,k} \right) c_{k,j} =$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} a_{i,l} b_{l,k} c_{k,j} = \sum_{l=1}^{n} \sum_{k=1}^{p} a_{i,l} b_{l,k} c_{k,j} =$$

$$= \sum_{l=1}^{n} a_{i,l} \left(\sum_{k=1}^{p} b_{l,k} c_{k,j} \right) = \sum_{l=1}^{n} a_{i,l} (\boldsymbol{BC})_{l,j} = (\boldsymbol{A}(\boldsymbol{BC}))_{i,j}$$

$m = q = i = j = 1,\ n = p = 2:$

		0	1	6
		2	3	7
4	5	10	19	193

		6
		7
0	1	7
2	3	33
4	5	193

$$193 = 10 \cdot 6 + 19 \cdot 7 = (4 \cdot 0 + 5 \cdot 2) \cdot 6 + (4 \cdot 1 + 5 \cdot 3) \cdot 7 =$$

$$(4 \cdot 0 \cdot 6 + 5 \cdot 2 \cdot 6) + (4 \cdot 1 \cdot 7 + 5 \cdot 3 \cdot 7) = (4 \cdot 0 \cdot 6 + 4 \cdot 1 \cdot 7) + (5 \cdot 2 \cdot 6 + 5 \cdot 3 \cdot 7) =$$

$$4 \cdot (0 \cdot 6 + 1 \cdot 7) + 5 \cdot (2 \cdot 6 + 3 \cdot 7) = 4 \cdot 7 + 5 \cdot 33 = 193$$

Efficiency of the product evaluation

$\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{B} \in \mathbb{R}^{n\times p}, \boldsymbol{C} \in \mathbb{R}^{p\times q} : (\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$

	$\boldsymbol{B}$	$\boldsymbol{C}$
$\boldsymbol{A}$	$\boldsymbol{AB}$	$(\boldsymbol{AB})\boldsymbol{C}$

$\boldsymbol{AB} \approx mnp$ arithmetic operations,
$(\boldsymbol{AB})\boldsymbol{C} \approx mpq$ operations,
in total $\approx mp(n+q)$ operations.

	$\boldsymbol{C}$
$\boldsymbol{B}$	$\boldsymbol{BC}$
$\boldsymbol{A}$	$\boldsymbol{A}(\boldsymbol{BC})$

$\boldsymbol{BC} \approx npq$ operations,
$\boldsymbol{A}(\boldsymbol{BC}) \approx mnq$ operations,
in total $\approx nq(m+p)$ operations.

When $m \ll n, p, q$, we get $mp(n+q) \ll nq(m+p)$.

					4
		2	−3	3	1
		−1	2	−1	−1
1	2	0	1	1	0
−2	−4	0	−2	−2	0

			4
			1
			−1
2	−3	3	2
−1	2	−1	−1
	1	2	0
	−2	−4	0

The first way has *more* intermediate values than the second.

Though the final result is the same in both ways, a suitable order of evaluation may affect the overall computational complexity.

Elementary matrices

Observation: Let $\boldsymbol{B}$ be the matix obtained from $\boldsymbol{A}$ by:

- the multiplication of the i-th row by $t \neq 0$. Then $\boldsymbol{B} = \boldsymbol{EA}$ where $e_{i,i} = t$, $e_{k,k} = 1$ for $k \neq i$ and $e_{k,l} = 0$ for $k \neq l$.
- the adding the j-th row to the i-th. Then $\boldsymbol{B} = \boldsymbol{EA}$ where $e_{i,j} = 1$, $e_{k,k} = 1$, for all k, and $e_{k,l} = 0$ for $i \neq k \neq l \neq j$.

Such matrices $\boldsymbol{E}$ are called *elementary* matrices.

Example:

				3	6	...	$a_{1,k}$	
				7	1	...	$a_{2,k}$	$i = 2$
				2	4	...	$a_{3,k}$	
				5	3	...	$a_{4,k}$	$j = 4$
1	0	0	0	3	6	...	$a_{1,k}$	
0	4	0	0	28	4	...	$4 \cdot a_{2,k}$	$t = 4$
0	0	1	0	2	4	...	$a_{3,k}$	
0	0	0	1	5	3	...	$a_{4,k}$	

1	0	0	0	3	6	...	$a_{1,k}$
0	1	0	1	12	4	...	$a_{2,k} + a_{4,k}$
0	0	1	0	2	4	...	$a_{3,k}$
0	0	0	1	5	3	...	$a_{4,k}$

Inverse matrix

Definition: If for $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ there exists $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ such that $\boldsymbol{AB} = \boldsymbol{I}_n$, then $\boldsymbol{B}$ is called the *inverse matrix* and denoted by $\boldsymbol{A}^{-1}$. If $\boldsymbol{A}$ has an inverse, then it is called *regular*, otherwise it is *singular*.

Theorem: For $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ the following are equivalent:

1. $\boldsymbol{A}$ is regular, i.e. $\exists \boldsymbol{B} \in \mathbb{R}^{n \times n} : \boldsymbol{AB} = \boldsymbol{I}_n$.
2. $\text{rank}(\boldsymbol{A}) = n$.
3. $\boldsymbol{A} \sim\sim \boldsymbol{I}_n$.
4. The system $\boldsymbol{Ax} = \mathbf{0}$ has only trivial solution $\boldsymbol{x} = \mathbf{0}$.

Proof: 2. $\Leftrightarrow$ 4. was already proved.
2. $\Rightarrow$ 3. by Gauss-Jordan elimination, 2. $\Leftarrow$ 3. trivially.
2. $\Rightarrow$ 1. Denote $\boldsymbol{I}_n = (\boldsymbol{e}^1 | \dots | \boldsymbol{e}^n)$. For $i = 1, \dots, n$ consider systems $\boldsymbol{Ax}^i = \boldsymbol{e}^i$. Since $\text{rank}(\boldsymbol{A}) = n$ we get $\boldsymbol{B} = (\boldsymbol{x}^1 | \dots | \boldsymbol{x}^n)$.
1. $\Rightarrow$ 2. If $\text{rank}(\boldsymbol{A}) < n$ then for some i, the i-th row of $\boldsymbol{A}$ can be eliminated by the other rows, thus $\boldsymbol{Ax}^i = \boldsymbol{e}^i$ has no solution as the only 1 in $\boldsymbol{e}^i$ cannot be eliminated by zeroes.

Inverse matrix

Corollary: If the inverse matrix $\boldsymbol{A}^{-1}$ exists, it is unique.

Theorem: The inverse matrix satisfies: $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_n$.

Proof: We first show that $\boldsymbol{A}^{-1}$ is regular:
If $\boldsymbol{A}^{-1}\boldsymbol{x} = \mathbf{0}$ has a solution then $\boldsymbol{x} = \boldsymbol{I}_n\boldsymbol{x} = \boldsymbol{A}\boldsymbol{A}^{-1}\boldsymbol{x} = \boldsymbol{A}\mathbf{0} = \mathbf{0}$.
Hence there exists $(\boldsymbol{A}^{-1})^{-1}$ and we get:
$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{I}_n = (\boldsymbol{A}^{-1}\boldsymbol{A})(\boldsymbol{A}^{-1}(\boldsymbol{A}^{-1})^{-1}) =$
$\boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{A}^{-1})(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}^{-1}\boldsymbol{I}_n(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}^{-1}(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{I}_n$

Corollary: If $\boldsymbol{B}\boldsymbol{A} = \boldsymbol{I}_n$ then $\boldsymbol{A}^{-1} = \boldsymbol{B}$.

Proof: $\boldsymbol{B}\boldsymbol{A} = \boldsymbol{I}_n \implies \boldsymbol{A}\boldsymbol{B} = \boldsymbol{I}_n \implies \boldsymbol{A}^{-1} = \boldsymbol{B}$

Inverse matrix calculation

- Assemble $(\boldsymbol{A}|\boldsymbol{I}_n)$ and by Gauss-Jordan elimination get $(\boldsymbol{I}_n|\boldsymbol{B})$.
- If this process fails, then $\boldsymbol{A}$ is singular.
- Denote $\boldsymbol{E}_1, \dots, \boldsymbol{E}_k$ the elementary matrices of the applied elementary transforms. Then the left side of $(\boldsymbol{A}|\boldsymbol{I}_n) \sim\sim (\boldsymbol{I}_n|\boldsymbol{B})$ yields $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n$, the right side yields $\boldsymbol{E}_k \cdots \boldsymbol{E}_1 \boldsymbol{I}_n = \boldsymbol{B}$ thus $\boldsymbol{B}\boldsymbol{A} = \boldsymbol{I}_n$ and therefore $\boldsymbol{A}^{-1} = \boldsymbol{B}$.
  (Indeed the columns of $\boldsymbol{B}$ are solutions of $\boldsymbol{A}\boldsymbol{x}^i = \boldsymbol{e}^i$)

Example:

$$(\boldsymbol{A}|\boldsymbol{I}_n) = \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 5 & 6 & 3 & -1 & 0 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & -1 & -5 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -8 \\ 0 & 1 & 0 & -3 & 1 & 6 \\ 0 & 0 & 1 & 3 & -1 & -5 \end{array}\right) = (\boldsymbol{I}_n|\boldsymbol{A}^{-1})$$

Correctness test: $$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 4 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 & -8 \\ -3 & 1 & 6 \\ 3 & -1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix equations

Observation: If $\boldsymbol{R}$ is regular, then: $\boldsymbol{A} = \boldsymbol{B} \iff \boldsymbol{AR} = \boldsymbol{BR}$.

Proof: $\Rightarrow$ trivially, $\Leftarrow$: $\boldsymbol{A} = \boldsymbol{AI}_n = \boldsymbol{ARR}^{-1} = \boldsymbol{BRR}^{-1} = \boldsymbol{B}$.

Exercises: Prove that regular matrices $\boldsymbol{A}, \boldsymbol{B}$ of the same order satisfy:

- $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
- $\boldsymbol{AB}$ is regular
- $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$

Matrix equations

$$\begin{aligned}
\boldsymbol{A}+\boldsymbol{X}=\boldsymbol{B} &\quad\Longrightarrow\quad \boldsymbol{X}=\boldsymbol{B}-\boldsymbol{A}=\boldsymbol{B}+(-1)\boldsymbol{A}\\
a\boldsymbol{X}=\boldsymbol{B} &\quad\Longrightarrow\quad \boldsymbol{X}=\tfrac{1}{a}\boldsymbol{B}\\
\boldsymbol{A}\boldsymbol{X}=\boldsymbol{B} &\quad\Longrightarrow\quad \boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B},\ \text{for regular } \boldsymbol{A}\\
\boldsymbol{X}\boldsymbol{A}=\boldsymbol{B} &\quad\Longrightarrow\quad \boldsymbol{X}=\boldsymbol{B}\boldsymbol{A}^{-1},\ \text{for regular } \boldsymbol{A}
\end{aligned}$$

Beware, products $\boldsymbol{A}^{-1}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}^{-1}$ could be distinct.

Equations $\begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$ and $\boldsymbol{X}' \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$

have different solutions $\boldsymbol{X} = \begin{pmatrix} -9 & 3 \\ 42 & -14 \end{pmatrix}$ and $\boldsymbol{X}' = \begin{pmatrix} 7 & -15 \\ 14 & -30 \end{pmatrix}$

Test:

		-9	3
		42	-14
9	2	3	-1
4	1	6	-2

		9	2
		4	1
7	-15	3	-1
14	-30	6	-2