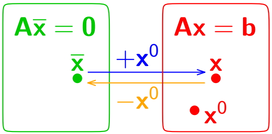
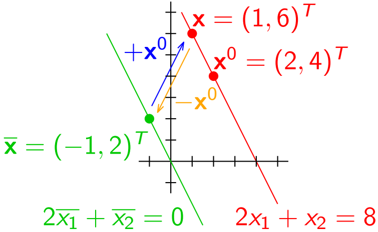
Homogeneous and non-homogeneous systems

Notation: Vectors of the same length are added by components.
We multiply a vector by a real number also by components.

Observation: If $\boldsymbol{x}$ and $\boldsymbol{x}^0$ are two solutions of $\boldsymbol{Ax} = \boldsymbol{b}$
then $\overline{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}^0$ is a solution of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$.





Proof: $\boldsymbol{A}\overline{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}^0) \overset{*}{=} \boldsymbol{Ax} - \boldsymbol{Ax}^0 = \boldsymbol{b} - \boldsymbol{b} = \mathbf{0}$.

$*$: $a_{i,1}(x_1 - x_1^0) + \cdots + a_{i,n}(x_n - x_n^0) = (a_{i,1}x_1 + \cdots + a_{i,n}x_n) - (a_{i,1}x_1^0 + \cdots + a_{i,n}x_n^0)$

Example: $2(-1) + 2 = 2(1-2) + (6-4) \overset{*}{=} (2 \cdot 1 + 6) - (2 \cdot 2 + 4) = 8 - 8 = 0$

Observation: If $\boldsymbol{x}^0$ is a solution of $\boldsymbol{Ax} = \boldsymbol{b}$ and $\overline{\boldsymbol{x}}$ is a solution of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$ then $\boldsymbol{x} = \overline{\boldsymbol{x}} + \boldsymbol{x}^0$ is a solution of $\boldsymbol{Ax} = \boldsymbol{b}$.

Proof: Analogously: $\boldsymbol{Ax} = \boldsymbol{A}(\overline{\boldsymbol{x}} + \boldsymbol{x}^0) = \boldsymbol{A}\overline{\boldsymbol{x}} + \boldsymbol{Ax}^0 = \boldsymbol{b} + \mathbf{0} = \boldsymbol{b}$.

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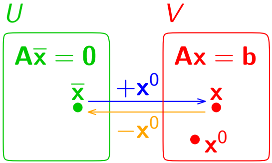
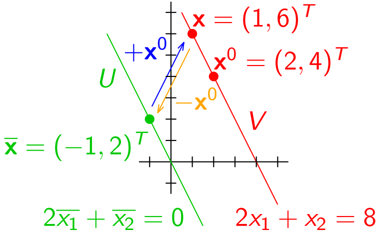




Theorem: Let $\boldsymbol{x}^0$ satisfy $\boldsymbol{A}\boldsymbol{x}^0 = \boldsymbol{b}$. Then the map $\bar{\boldsymbol{x}} \to \bar{\boldsymbol{x}} + \boldsymbol{x}^0$ is a bijection between the sets $\{\bar{\boldsymbol{x}} \colon \boldsymbol{A}\bar{\boldsymbol{x}} = \mathbf{0}\}$ and $\{\boldsymbol{x} \colon \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$.

Proof: Denote $U = \{\bar{\boldsymbol{x}} \colon \boldsymbol{A}\bar{\boldsymbol{x}} = \mathbf{0}\}$, $V = \{\boldsymbol{x} \colon \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$,
$f \colon U \to V$ by $f(\bar{\boldsymbol{x}}) = \bar{\boldsymbol{x}} + \boldsymbol{x}^0$, and $g \colon V \to U$ by $g(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}^0$.

$$\left.\begin{array}{l} g \circ f \text{ is the identity on } U \Longrightarrow f \text{ is injective} \\ f \circ g \text{ is the identity on } V \Longrightarrow f \text{ is surjective} \end{array}\right\} \Longrightarrow f \text{ is bijective.}$$

Solutions of homogeneous systems $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$

Theorem: For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ a matrix of rank r all solutions of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$ can be written as $\overline{\boldsymbol{x}} = p_1\overline{\boldsymbol{x}}^1 + p_2\overline{\boldsymbol{x}}^2 + \cdots + p_{n-r}\overline{\boldsymbol{x}}^{n-r}$, where

- $p_1, \ldots, p_{n-r}$ are arbitrary real parameters and
- $\overline{\boldsymbol{x}}^1, \ldots, \overline{\boldsymbol{x}}^{n-r}$ are suitable solutions of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$.

The system has only the trivial solution $\overline{\boldsymbol{x}} = \mathbf{0}$ iff $\operatorname{rank}(\boldsymbol{A}) = n$.

Example:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \qquad \begin{aligned} \overline{x_1} &= -4\overline{x_2} - 3\overline{x_3} - 2\overline{x_4} - \overline{x_5} &&= -4\overline{x_2} + 4\overline{x_4} \\ \overline{x_3} &= -2\overline{x_4} - \overline{x_5} &&= -2\overline{x_4} \\ \overline{x_5} &= 0 && \end{aligned}$$

When we rename $\overline{x_2}$ and $\overline{x_4}$ by parameters p_1 and p_2, we get:

$$\begin{aligned} \overline{x_1} &= -4\overline{x_2} + 4\overline{x_4} = -4p_1 + 4p_2 \\ \overline{x_2} &= p_1 \\ \overline{x_3} &= -2\overline{x_4} = -2p_2, \\ \overline{x_4} &= p_2 \\ \overline{x_5} &= 0 \end{aligned} \qquad \text{that is } \overline{\boldsymbol{x}} = p_1\overline{\boldsymbol{x}}^1 + p_2\overline{\boldsymbol{x}}^2 = p_1 \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} 4 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

Solutions of homogeneous systems $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$

Theorem: For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ a matrix of rank r all solutions of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$ can be written as $\overline{\boldsymbol{x}} = p_1\overline{\boldsymbol{x}}^1 + p_2\overline{\boldsymbol{x}}^2 + \cdots + p_{n-r}\overline{\boldsymbol{x}}^{n-r}$, where

- $p_1, \ldots, p_{n-r}$ are arbitrary real parameters and
- $\overline{\boldsymbol{x}}^1, \ldots, \overline{\boldsymbol{x}}^{n-r}$ are suitable solutions of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$.

The system has only the trivial solution $\overline{\boldsymbol{x}} = \mathbf{0}$ iff $\text{rank}(\boldsymbol{A}) = n$.

Proof: Rename the free variables as $p_1, \ldots, p_{n-r}$.

By the backward substitution we can express each component of the solution as a linear function of the free variables, i.e.

$$\begin{aligned} \overline{x_1} &= \alpha_{1,1}p_1 + \cdots + \alpha_{1,n-r}p_{n-r} \\ &\vdots \\ \overline{x_n} &= \alpha_{n,1}p_1 + \cdots + \alpha_{n,n-r}p_{n-r} \end{aligned}$$

Choose $\overline{\boldsymbol{x}}^1 = (\alpha_{1,1}, \ldots, \alpha_{n,1})^T, \ldots, \overline{\boldsymbol{x}}^{n-r} = (\alpha_{1,n-r}, \ldots, \alpha_{n,n-r})^T$.

These solve $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$ as any such $\overline{\boldsymbol{x}}^i$ comes from: $p_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

If $\text{rank}(\boldsymbol{A}) = n$, all variables are leading, and $\mathbf{0}$ is the only solution.

Solutions of homogeneous systems $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$

Theorem: For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ a matrix of rank r all solutions of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$ can be written as $\overline{\boldsymbol{x}} = p_1\overline{\boldsymbol{x}}^1 + p_2\overline{\boldsymbol{x}}^2 + \cdots + p_{n-r}\overline{\boldsymbol{x}}^{n-r}$, where

- $p_1, \ldots, p_{n-r}$ are arbitrary real parameters and
- $\overline{\boldsymbol{x}}^1, \ldots, \overline{\boldsymbol{x}}^{n-r}$ are suitable solutions of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$.

The system has only the trivial solution $\overline{\boldsymbol{x}} = \mathbf{0}$ iff $\text{rank}(\boldsymbol{A}) = n$.

Corollary for *non-homogeneous* systems:

Let a system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a nonempty set of solutions where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is a matrix of rank r. Then all solutions of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ can be written as $\boldsymbol{x} = \boldsymbol{x}^0 + p_1\overline{\boldsymbol{x}}^1 + p_2\overline{\boldsymbol{x}}^2 + \cdots + p_{n-r}\overline{\boldsymbol{x}}^{n-r}$, where

- $p_1, \ldots, p_{n-r}$ are arbitrary real parameters,
- $\overline{\boldsymbol{x}}^1, \ldots, \overline{\boldsymbol{x}}^{n-r}$ are suitable solutions of $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$, and
- $\boldsymbol{x}^0$ is an arbitrary solution of the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Solutions of systems $\boldsymbol{Ax} = \boldsymbol{b}$ — summary and example

1. Transform the augmented matrix $(\boldsymbol{A}|\boldsymbol{b})$ into the echelon form:

$$(\boldsymbol{A}|\boldsymbol{b}) = \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 2 & 8 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 6 & 9 & 5 \\ 2 & 8 & 7 & 6 & 3 & 3 \end{array}\right) \sim\sim \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = (\boldsymbol{A}'|\boldsymbol{b}')$$

2. If a pivot is in the last column, then no solution exists.
3. Otherwise solve first the homogeneous system $\boldsymbol{A}'\overline{\boldsymbol{x}} = \mathbf{0}$, i.e.

$$\begin{aligned} \overline{x_5} &= 0 \\ \overline{x_3} &= -2\overline{x_4} - \overline{x_5} &&= -2\overline{x_4} \\ \overline{x_1} &= -4\overline{x_2} - 3\overline{x_3} - 2\overline{x_4} - \overline{x_5} &&= -4\overline{x_2} + 4\overline{x_4} \end{aligned}$$

4. Replace the free variables in $\overline{\boldsymbol{x}}$ with parameters:

$$\overline{\boldsymbol{x}} = p_1(-4, 1, 0, 0, 0)^T + p_2(4, 0, -2, 1, 0)^T$$

5. Finally, find any solution of the non-homogeneous system $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$, e.g. $\boldsymbol{x}^0 = \left(4, -1, 0, \frac{1}{3}, \frac{1}{3}\right)^T$ and get:

$$\boldsymbol{x} = \left(4, -1, 0, \frac{1}{3}, \frac{1}{3}\right)^T + p_1(-4, 1, 0, 0, 0)^T + p_2(4, 0, -2, 1, 0)^T$$

Verification

Substitute $\boldsymbol{x}$ *including parameters* into the *original* system $\boldsymbol{Ax} = \boldsymbol{b}$, ie., verify $\boldsymbol{x}^0$ for $\boldsymbol{Ax} = \boldsymbol{b}$ and also $\overline{\boldsymbol{x}}^1, \ldots, \overline{\boldsymbol{x}}^{n-r}$ for $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$.

Example:

For a system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 &= 1 \\ 2x_1 + 8x_2 + 4x_3 &= 0 \\ 3x_3 + 6x_4 + 9x_5 &= 5 \\ 2x_1 + 8x_2 + 7x_3 + 6x_4 + 3x_5 &= 3 \end{aligned}$$

substitute

$$\boldsymbol{x} = \left(4, -1, 0, \tfrac{1}{3}, \tfrac{1}{3}\right)^T + p_1(-4, 1, 0, 0, 0)^T + p_2(4, 0, -2, 1, 0)^T$$

$$= \left(4 - 4p_1 + 4p_2, -1 + p_1, -2p_2, \tfrac{1}{3} + p_2, \tfrac{1}{3}\right)^T$$ as follows:

$$\begin{aligned} (4 - 4p_1 + 4p_2) + 4(p_1 - 1) + 3(-2p_2) + 2(\tfrac{1}{3} + p_2) + \tfrac{1}{3} &= 1 \\ 2(4 - 4p_1 + 4p_2) + 8(p_1 - 1) + 4(-2p_2) &= 0 \\ 3(-2p_2) + 6(\tfrac{1}{3} + p_2) + 9 \cdot \tfrac{1}{3} &= 5 \\ 2(4 - 4p_1 + 4p_2) + 8(p_1 - 1) + 7(-2p_2) + 6(\tfrac{1}{3} + p_2) + 3 \cdot \tfrac{1}{3} &= 3 \end{aligned}$$

Observe that the parameters on the left side will cancel each other.

What the tests verify

If we do not insert the solution $\boldsymbol{x}$ including the parameters into the original system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, we may not detect the following errors:

- Without the parameters we verify only the solution $\boldsymbol{x}^0$.
- A specific choice of parameters verifies only one solution, not all. We wouldn't have to reveal faulty solutions.

Example: We check $\boldsymbol{x} = (1,3)^T + p_1(1,1)^T$ for $2x_1 + x_2 = 8$. For $p_1 = 1$, we have $\boldsymbol{x} = (2,4)^T$, which satisfies $2 \cdot 2 + 4 = 8$, but neither $\boldsymbol{x}^0 = (1,3)^T$ nor choices of p_1 are valid solutions.

- Substitution into $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$ verifies the correctness of the backward substitution, but not of Gaussian elimination.

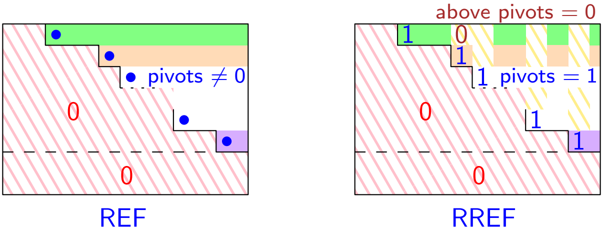
Warning: We cannot yet verify the completeness of the solution set. It may happen that we add a new condition due to an error in Gaussian elimination. Then we get only a *subset* of all solutions.

E.g. on error in the last column we may not obtain any solution:

$$\left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & -4 & -2 & -2 \\ 0 & 0 & 3 & 6 & 9 & 5 \end{array}\right) \underset{+2\mathrm{II}}{\sim} \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 6 & 9 & 5 \end{array}\right) \checkmark \text{ vs. } \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & -4 & -2 & -2 \\ 0 & 0 & 3 & 6 & 9 & 5 \end{array}\right) \underset{+2\mathrm{II}}{\sim} \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 3 & 6 & 9 & 5 \end{array}\right) \times$$

Reduced row echelon form

Definition: A row echelon form of a matrix is *reduced* if each pivot is 1 and all other elements in pivot columns are 0es.



Reduced row echelon form

Definition: A row echelon form of a matrix is *reduced* if each pivot is 1 and all other elements in pivot columns are 0es.

Fact: Every matrix $\boldsymbol{A}$ has a unique $\boldsymbol{A}'$ in reduced row echelon form such that $\boldsymbol{A} \sim\sim \boldsymbol{A}'$. "Pf": distinct r.r.e.f. yields different solutions.

Any matrix in row echelon form can be reduced by:

- Divide each row by $a_{i,j(i)}$, thus get 1s as pivots.
- Foreach $i = r, \ldots, 1$, eliminate each $a_{i',j(i)}$ with $i' < i$ by addition $-a_{i',j(i)}$-multiple of the i-th row to i'-th row. thus to get 0es above the pivot $a_{i,j(i)}$

Example:

$$\left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right) \underset{:6}{\sim} \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right) \begin{array}{l} -\mathrm{III} \\ -\mathrm{III} \\ \end{array} \sim$$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right) \overset{-3\mathrm{II}}{\sim} \left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

This process is sometimes called the *Gauss-Jordan elimination*.

Advantages of reduced row echelon form

$$\boldsymbol{x} = \left(-\tfrac{4}{3}, 0, \tfrac{2}{3}, 0, \tfrac{1}{3}\right)^T + p_1(-4,1,0,0,0)^T + p_2(4,0,-2,1,0)^T$$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right) \sim\sim \left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

yields directly the solution
$\boldsymbol{x}^0 = \left(-\frac{4}{3}, 0, \frac{2}{3}, 0, \frac{1}{3}\right)^T$
by choosing $x_2 = x_4 = 0$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

solves directly $\boldsymbol{Ax} = \mathbf{0}$ as
$\overline{\boldsymbol{x}}^1 = (-4, 1, 0, 0, 0)^T$,
i.e. for $x_2 = 1$ and $x_4 = 0$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

solves directly $\boldsymbol{Ax} = \mathbf{0}$ as
$\overline{\boldsymbol{x}}^2 = (4, 0, -2, 1, 0)^T$,
i.e. for $x_2 = 0$ and $x_4 = 1$

Ill-conditioned system

$$\begin{aligned} 667x_1 - 835x_2 &= 168 \\ 266x_1 - 333x_2 &= \mathbf{67} \end{aligned}$$

solution $\boldsymbol{x} = (-1, -1)^T$

$$\begin{aligned} 667x_1 - 835x_2 &= 168 \\ 266x_1 - 333x_2 &= \mathbf{68} \end{aligned}$$

solution $\boldsymbol{x} = (834, 666)^T$