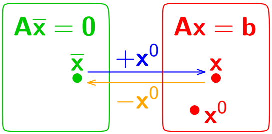
Homogeneous and non-homogeneous systems

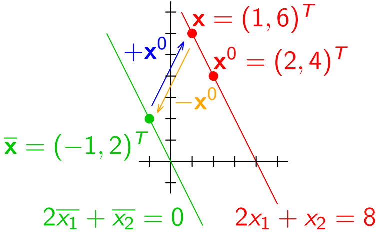
Observation: If $\boldsymbol{x}$ and $\boldsymbol{x}^0$ are two solutions of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ then $\overline{\mathbf{x}} = \boldsymbol{x} - \boldsymbol{x}^0$ is a solution of $\boldsymbol{A}\overline{\mathbf{x}} = \mathbf{0}$.

Proof: $\boldsymbol{A}\overline{\mathbf{x}} = \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}^0) = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{x}^0 = \boldsymbol{b} - \boldsymbol{b} = \mathbf{0}$.

Observation: If $\boldsymbol{x}^0$ is a solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\overline{\mathbf{x}}$ is a solution of $\boldsymbol{A}\overline{\mathbf{x}} = \mathbf{0}$ then $\boldsymbol{x} = \overline{\mathbf{x}} + \boldsymbol{x}^0$ is a solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Proof: Analogously: $\boldsymbol{A}\boldsymbol{x} = \cdots = \boldsymbol{b}$ (add details yourself).





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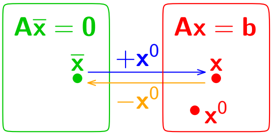

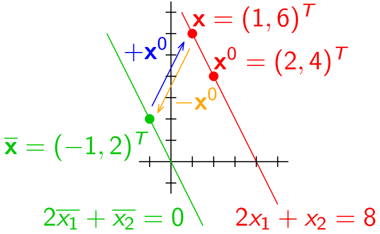

Theorem: Let $\boldsymbol{x}^0$ satisfy $\boldsymbol{A}\boldsymbol{x}^0 = \boldsymbol{b}$. Then the map $\bar{\mathbf{x}} \to \bar{\mathbf{x}} + \boldsymbol{x}^0$ is a bijection between the sets $\{\bar{\mathbf{x}} \colon \boldsymbol{A}\bar{\mathbf{x}} = \mathbf{0}\}$ and $\{\boldsymbol{x} \colon \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$.

Proof: Denote $U = \{\bar{\mathbf{x}} \colon \boldsymbol{A}\bar{\mathbf{x}} = \mathbf{0}\}$, $V = \{\boldsymbol{x} \colon \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$,
$f \colon U \to V$ by $f(\bar{\mathbf{x}}) = \bar{\mathbf{x}} + \boldsymbol{x}^0$, and $g \colon V \to U$ by $g(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}^0$.

$$\left.\begin{array}{l} g \circ f \text{ is the identity on } U \Longrightarrow f \text{ is injective} \\ f \circ g \text{ is the identity on } V \Longrightarrow f \text{ is surjective} \end{array}\right\} \Longrightarrow f \text{ is bijective.}$$

Solutions of homogeneous systems

Theorem: Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a matrix of rank r. Then all solutions of $\boldsymbol{A}\overline{\mathbf{x}} = \mathbf{0}$ can be described as $\overline{\mathbf{x}} = p_1 \overline{\mathbf{x}}^1 + p_2 \overline{\mathbf{x}}^2 + \cdots + p_{n-r} \overline{\mathbf{x}}^{n-r}$, where $p_1, \dots, p_{n-r}$ are arbitrary real parameters and $\overline{\mathbf{x}}^1, \dots, \overline{\mathbf{x}}^{n-r}$ are suitable solutions of $\boldsymbol{A}\overline{\mathbf{x}} = \mathbf{0}$. The system has only the trivial solution $\overline{\mathbf{x}} = \mathbf{0}$ if and only if $\operatorname{rank}(\boldsymbol{A}) = n$.

Proof: Rename the free variables as $p_1, \dots, p_{n-r}$.
By the backward substitution we can express each component of the solution as a linear function of the free variables, i.e.

$$\begin{aligned} \overline{x_1} &= \alpha_{1,1} p_1 + \cdots + \alpha_{1,n-r} p_{n-r} \\ &\vdots \\ \overline{x_n} &= \alpha_{n,1} p_1 + \cdots + \alpha_{n,n-r} p_{n-r} \end{aligned}$$

Choose $\overline{\mathbf{x}}^1 = (\alpha_{1,1}, \dots, \alpha_{n,1})^T, \dots, \overline{\mathbf{x}}^{n-r} = (\alpha_{1,n-r}, \dots, \alpha_{n,n-r})^T$.

These solve $\boldsymbol{A}\overline{\mathbf{x}} = \mathbf{0}$ as any such $\overline{\mathbf{x}}^i$ comes from: $p_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

If $\operatorname{rank}(\boldsymbol{A}) = n$, all variables are leading, and $\mathbf{0}$ is the only solution.

Solutions of non-homogeneous systems

Corollary: Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a matrix of rank r. If the system $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{x}^0$, then all solutions of $\boldsymbol{Ax} = \boldsymbol{b}$ can be described as $\boldsymbol{x} = \boldsymbol{x}^0 + p_1 \bar{\mathbf{x}}^1 + p_2 \bar{\mathbf{x}}^2 + \cdots + p_{n-r} \bar{\mathbf{x}}^{n-r}$, where $p_1, \ldots, p_{n-r}$ are arbitrary real parameters, and $\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^{n-r}$ are suitable solutions of $\boldsymbol{A}\bar{\mathbf{x}} = \mathbf{0}$.

Summary:

1. Transform the augmented matrix $(\boldsymbol{A}|\boldsymbol{b})$ into the echelon form:

$$(\boldsymbol{A}'|\boldsymbol{b}') = \left(\begin{array}{ccccc|c} -1 & 2 & -4 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & -4 \end{array}\right)$$

2. If a pivot is in the last column, then no solution exists.

3. Otherwise solve first the homogeneous system $\boldsymbol{A}'\overline{\boldsymbol{x}} = \mathbf{0}$, i.e.

$$\begin{pmatrix} -1 & 2 & -4 & 0 & -3 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \overline{\mathbf{x}} = \mathbf{0}$$

$$\begin{aligned} \overline{x_5} &= 0 & & \\ \overline{x_3} &= -3\overline{x_4} - 2\overline{x_5} &&= -3\overline{x_4} \\ \overline{x_1} &= 2\overline{x_2} - 4\overline{x_3} - 3\overline{x_5} &&= 2\overline{x_2} + 12\overline{x_4} \end{aligned}$$

With parameters p_1 and p_2 (substitutes for $\overline{x_2}$ and $\overline{x_4}$):

$$\begin{aligned} \overline{x_1} &= 2p_1 + 12p_2 \\ \overline{x_2} &= p_1 \\ \overline{x_3} &= -3p_2 \\ \overline{x_4} &= p_2 \\ \overline{x_5} &= 0 \end{aligned} \qquad \text{equivalently } \overline{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} p_1 + \begin{pmatrix} 12 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} p_2$$

4. Finally, find any solution of the non-homogeneous system $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$, e.g. $\boldsymbol{x}^0 = (-18, 0, 6, 0, -2)^T$ and get:

$$\boldsymbol{x} = (-18, 0, 6, 0, -2)^T + p_1(2, 1, 0, 0, 0)^T + p_2(12, 0, -3, 1, 0)^T$$

Reduced row echelon form

Definition: A row echelon form of a matrix is *reduced* if each pivot is 1 and all other elements in pivot columns are 0es.

Fact: Every matrix $\boldsymbol{A}$ has a unique $\boldsymbol{A}'$ in reduced row echelon form such that $\boldsymbol{A} \sim\sim \boldsymbol{A}'$. "Pf": distinct r.r.e.f. yields different solutions.

Any matrix in row echelon form can be reduced by:

- Multiply each row by $a_{i,j(i)}^{-1}$, thus get 1s as pivots.
- Foreach $i = r, \ldots, 1$, eliminate each $a_{i',j(i)}$ with $i' < i$ by addition $-a_{i',j(i)}$-multiple of the i-th row to i'-th row. thus to get 0es above the pivot $a_{i,j(i)}$

Example:

$$\left(\begin{array}{ccccc|c} -1 & 2 & -4 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & -4 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -2 & 4 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 4 & 0 & 0 & 6 \\ 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -2 & 0 & -12 & 0 & -18 \\ 0 & 0 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right) \sim$$

This process is sometimes called the *Gauss-Jordan elimination*.

Advantages of reduced row echelon form

$$\boldsymbol{x} = (-18, 0, 6, 0, -2)^T + p_1(2, 1, 0, 0, 0)^T + p_2(12, 0, -3, 1, 0)^T$$

$$\left(\begin{array}{ccccc|c} -1 & 2 & -4 & 0 & -3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & -4 \end{array}\right) \sim\sim \left(\begin{array}{ccccc|c} 1 & -2 & 0 & -12 & 0 & -18 \\ 0 & 0 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & -12 & 0 & -18 \\ 0 & 0 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

yields directly the solution
$\boldsymbol{x}^0 = (-18, 0, 6, 0, -2)^T$
(by choosing $x_2 = x_4 = 0$)

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & -12 & 0 & -18 \\ 0 & 0 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

yields directly the vector
$(2, 1, 0, 0, 0)^T$
(by choosing $x_2 = p_1$)

$$\left(\begin{array}{ccccc|c} 1 & -2 & 0 & -12 & 0 & -18 \\ 0 & 0 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

yields directly the vector
$(12, 0, -3, 1, 0)^T$
(by choosing $x_4 = p_2$)

Ill-conditioned system

$$\begin{aligned} 667x_1 - 835x_2 &= 168 \\ 266x_1 - 333x_2 &= \mathbf{67} \end{aligned}$$

solution $\boldsymbol{x} = (-1, -1)^T$

$$\begin{aligned} 667x_1 - 835x_2 &= 168 \\ 266x_1 - 333x_2 &= \mathbf{68} \end{aligned}$$

solution $\boldsymbol{x} = (834, 666)^T$