Real vectors

Definition: A real *vector* $\boldsymbol{b}$ with m components is an ordered m-tuple of real numbers $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.
We write $\boldsymbol{b} \in \mathbb{R}^m$.

We consider *column* vectors.
For the row-wise notation
we use the transposition, i.e. $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = (b_1, b_2, \dots, b_m)^T$.

The vector $\mathbf{0} = (0, \dots, 0)^T \in \mathbb{R}^m$ is the *zero vector*.

An ordered n-tuple of variables $\boldsymbol{x} = (x_1, \dots, x_n)^T$
is the *vector of unknowns*.

Real matrices

Definition: A real *matrix* $\boldsymbol{A}$ of order $m \times n$ is a collection of mn real numbers arranged in an array with m rows and n columns.
We write $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.

$$\boldsymbol{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

The elements of a matrix are denoted as $(\boldsymbol{A})_{i,j} = a_{i,j}$, also as a_{ij}.

A *square matrix* has the same number of rows and columns.

Systems of equations

Definition: Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ is a vector of unknowns.

The *system of m linear equations in n unknowns* is $\boldsymbol{Ax} = \boldsymbol{b}$, in expanded form written as:

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \ldots & + & a_{1,n}x_n & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \ldots & + & a_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \ldots & + & a_{m,n}x_n & = & b_m \end{array}$$

The matrix $\boldsymbol{A}$ is is the *matrix of the system*,
the vector $\boldsymbol{b}$ is the *right-hand side vector*,
the matrix $(\boldsymbol{A}|\boldsymbol{b}) \in \mathbb{R}^{m \times (n+1)}$ is the *augmented matrix*.

A vector $\boldsymbol{x} \in \mathbb{R}^n$ is the *solution* of $\boldsymbol{Ax} = \boldsymbol{b}$ if it satisfies all m equations, i.e:

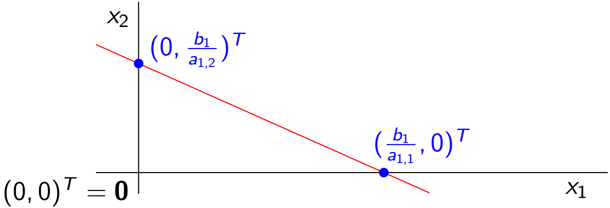
$$\forall i \in \{1, \ldots, m\} : a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n = b_i.$$

Systems $\boldsymbol{Ax} = \boldsymbol{0}$ are called *homogeneous* and always allow $\boldsymbol{x} = \boldsymbol{0}$.

Geometric meaning — one equation in two unknowns

$a_{1,1}x_1 + a_{1,2}x_2 = b_1$ or equivalently $\begin{pmatrix} a_{1,1} & a_{1,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \end{pmatrix}$

- If $a_{1,1} \neq 0$ or $a_{1,2} \neq 0$ then the set of solutions forms a line in the Euclidean plane.



- It could be parallel to one of the axes, e.g. to x_1, if $a_{1,1} = 0$.

Degenerate cases:

- If $a_{1,1} = a_{1,2} = 0$ and $b_1 \neq 0$, then the system has no solution.
- If $a_{1,1} = a_{1,2} = 0$ and $b_1 = 0$, then all points of the Euclidean plane are solutions.

Two equations in two unknowns

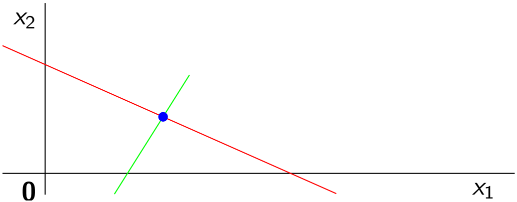
$$a_{1,1}x_1 + a_{1,2}x_2 = b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 = b_2$$

or

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

If both equations are nondegenerate, then the set of solutions is the intersection of two lines, which could be

- a point:



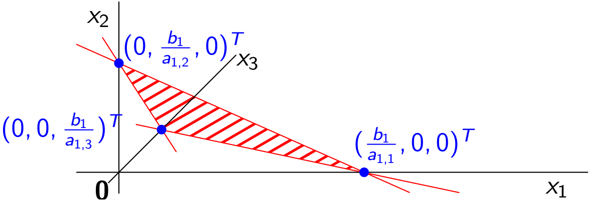
- an empty set, if the two lines are distinct parallel,
- a line, if the two lines are identical.

Among the degenerate cases, the system $\mathbf{0x} = \mathbf{0}$
also yields all points of the Euclidean plane as the set of solutions.

One equation in three unknowns

$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 = b_1$

- In the nondegenerate case $a_{1,1} \neq 0 \vee a_{1,2} \neq 0 \vee a_{1,3} \neq 0$ solutions form a plane in the 3-dimensional Euclidean space:



Degenerate cases:

- If $a_{1,1} = a_{1,2} = a_{1,3} = 0$ and $b_1 \neq 0$, then no solution exists.
- If $a_{1,1} = a_{1,2} = a_{1,3} = 0$ and $b_1 = 0$ then all points of the Euclidean space are solutions.

Elementary equivalent row transformations

Definition: We write $\boldsymbol{A} \sim \boldsymbol{A}'$ if $\boldsymbol{A}'$ is obtained from $\boldsymbol{A}$ by any of the following *elementary equivalent row transformations*:

1. Multiplication of the i-th row by a *nonzero* $t \in \mathbb{R} \setminus \{0\}$,
   formally: $a'_{k,l} = \begin{cases} a_{k,l} & \text{if } k \neq i \\ t a_{i,l} & \text{if } k = i \end{cases}$
2. Adding the j-th row to the i-th row,
   formally: $a'_{k,l} = \begin{cases} a_{k,l} & \text{if } k \neq i \\ a_{i,l} + a_{j,l} & \text{if } k = i \end{cases}$

From the two above the following two can be derived:

3. Adding the j-th row multiplied by $t \in \mathbb{R}$ to the i-th row.
4. Exchange of two rows.

A series of elementary transformations is denoted as $\boldsymbol{A} \sim\sim \boldsymbol{A}'$.

Use of elementary transformations

Theorem: Let $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$ be two systems such that $(\boldsymbol{A}|\boldsymbol{b}) \sim\sim (\boldsymbol{A}'|\boldsymbol{b}')$. Then both systems have identical solution sets.

Proof: It suffices to show that the solution set is preserved if a single transform of the first or of the second type is performed. Denote $X = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ and $X' = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'\}$.

The goal is to show that $X = X'$ (i.e. $\forall \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in X \Leftrightarrow \boldsymbol{x} \in X'$). We get the equality from two inclusions $X \subseteq X'$ and $X' \subseteq X$, i.e. $\forall \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in X \Rightarrow \boldsymbol{x} \in X'$ and $\forall \boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in X' \Rightarrow \boldsymbol{x} \in X$.

1a. to argue $X \subseteq X'$ for the i-th row scaling by $t \neq 0$:
As only the i-th row/equation is changed, any $\boldsymbol{x} \in X$ automatically satisfies the unchanged equations of $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$.
To get $\boldsymbol{x} \in X'$ we must verify the i-th equation of $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$.
Going from l.h.s. to r.h.s.: $a'_{i,1}x_1 + \cdots + a'_{i,n}x_n =$
$ta_{i,1}x_1 + \cdots + ta_{i,n}x_n = t(a_{i,1}x_1 + \cdots + a_{i,n}x_n) = tb_i = b'_i$
Green indicates the relationship between $(\boldsymbol{A}|\boldsymbol{b})$ and $(\boldsymbol{A}'|\boldsymbol{b}')$. i.e. the transform; red the assumption $\boldsymbol{x} \in X$, i.e. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Summary of cases a single elementary transformation of the first or of the second second type and the i-th equation:

1a. $\boldsymbol{Ax} = \boldsymbol{b} \Rightarrow \boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$: $a'_{i,1}x_1 + \cdots + a'_{i,n}x_n =$
$ta_{i,1}x_1 + \cdots + ta_{i,n}x_n = t(a_{i,1}x_1 + \cdots + a_{i,n}x_n) = tb_i = b'_i$

1b. $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}' \Rightarrow \boldsymbol{Ax} = \boldsymbol{b}$: $a_{i,1}x_1 + \cdots + a_{i,n}x_n =$
$\frac{1}{t}(ta_{i,1}x_1 + \cdots + ta_{i,n}x_n) = \frac{1}{t}(a'_{i,1}x_1 + \cdots + a'_{i,n}x_n) = \frac{1}{t}b'_i = \frac{1}{t}tb_i = b_i$

2a. $\boldsymbol{Ax} = \boldsymbol{b} \Rightarrow \boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$: $a'_{i,1}x_1 + \cdots + a'_{i,n}x_n =$
$(a_{i,1} + a_{j,1})x_1 + \cdots + (a_{i,n} + a_{j,n})x_n =$
$(a_{i,1}x_1 + \cdots + a_{i,n}x_n) + (a_{j,1}x_1 + \cdots + a_{j,n}x_n) = b_i + b_j = b'_i$

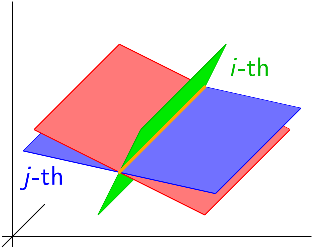
2b. $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}' \Rightarrow \boldsymbol{Ax} = \boldsymbol{b}$: $a_{i,1}x_1 + \cdots + a_{i,n}x_n =$
$a_{i,1}x_1 + \cdots + a_{i,n}x_n + b_j - b_j =$
$(a_{i,1}x_1 + \cdots + a_{i,n}x_n) + (a_{j,1}x_1 + \cdots + a_{j,n}x_n) - b_j =$
$(a_{i,1} + a_{j,1})x_1 + \cdots + (a_{i,n} + a_{j,n})x_n - b_j =$
$(a'_{i,1}x_1 + \cdots + a'_{i,n}x_n) - b_j = b'_i - b_j = b_i + b_j - b_j = b_i$

The color of $=$ means either the transform from $(\boldsymbol{A}|\boldsymbol{b})$ to $(\boldsymbol{A}'|\boldsymbol{b}')$, the assumption of the case, or an algebraic rearrangement of terms.

Geometric meaning of elementary transformations

1., 4. Multiplication of a row or swapping two rows does not change the position of the hyperplanes.

2., 3. Adding the j-th row (multiplied by $t \in \mathbb{R}$) to the i-th row moves the i-th hyperplane so that the intersection with the others remains unchanged.



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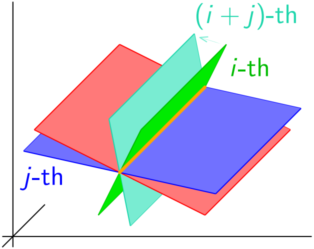

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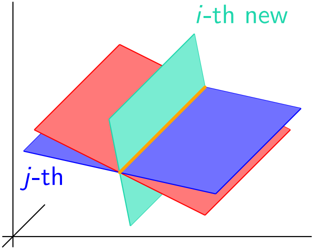

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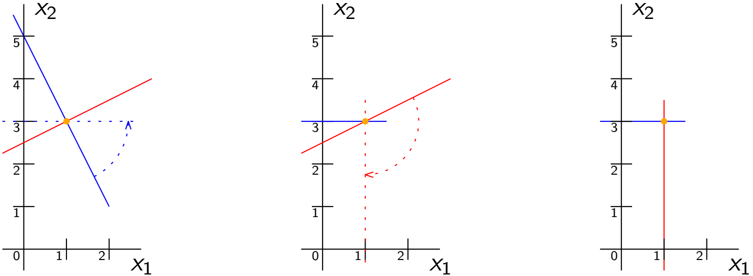
Goal: move the hyperplanes so the solution could be seen easily.



$$2x_1 + x_2 = 5$$
$$-x_1 + 2x_2 = 5$$
$+2\times,\ :3$

$$x_2 = 3$$
$$-x_1 + 2x_2 = 5$$
$-2\times,\ \cdot(-1)$

$$x_2 = 3$$
$$x_1 = 1$$