Where and how can linear algebra be useful in practice.

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All examples are simplified to demonstrate the use of tools of linear algebra. Real implementations are *much* more complex.

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### Geometric problems

Systems of linear equations with 2-3 unknowns correspond to finding the intersection of two lines; of a line and a plane, etc. Traditional and natural application include:

- surveying triangulations,
- descriptive and analytic geometry,
- astronomy parallaxes,
- 2D and 3D graphics, tomography, ...





The trigonometric network for the military mapping 1806–69 www.geolab.cz

Liu Hui (cca 225 — cca 295): Haidao Suanjing (The Sea Island Mathematical Manual) Cover from the 1726 reprint.

# Problems up to hundreds of variables

#### Circuit analysis



Question: What currents flow through branches of the circuit?

Units:

voltage: V

resistance:  $k\Omega$ 

current: mA

# Problems up to hundreds of variables

#### Circuit analysis



Kirchhoff laws

In any node, the entering currents saturate exactly the leaving currents.

In a loop, the sources exactly saturate appliances' voltages (as the  $I \cdot R$  terms).

### Problems up to hundreds of variables

#### Circuit analysis



Stigler's diet problem (1939), solved by Dantzig (1946)

The goal is to compile a diet of 77 foods to meet the recommended intake of 9 different nutrients while minimizing costs.

... leads to 9 inequalities on 77 variables, plus price optimization

# Weather forecast



For weather forecasting the temperature, pressure, humidity and the speed and direction of airflow should be determined with sufficient accuracy based on the current weather reports from weather stations.

Such situation could be described by a system of differential equations. Their exact solution is difficult and in some cases impossible.



For solving large systems of second order differential equations engineers use the *finite element method*.

It it based on a division of the investigated space into small cells. On each cell the desired function is approximated by a linear combination of "simple"functions, e.g. piecewise linear.







An approximation of a smooth function by a piecewise linear function. A decomposition of a piecewise linear function to a linear combination of simple functions. An example of a piecewise linear function on a triangulation in  $\mathbb{R}^2$ 

The ALADIN weather model from 2019 uses a three-dimensional grid over Central Europe covering area of about  $2800 \times 2200$  km (with a pitch of about 2.3 km) and 87 vertical levels. The total number of grid points and hence the number of the corresponding cells, is more than  $100\,000\,000$ .



The resulting given system of *differential* equations is converted to a system of order of billions of *linear* equations.

Meteorologists at the ČHMÚ in Komořany need and can solve such a system. The weather forecast is updated every 3 hours according to the calculated model.

# Image compression in jpeg

 $\frac{Jpeg}{group}$  (Joint photographic expert group) is a raster graphic format.

Its principle is as follows\*:

The image is first converted into the YCbCr color space. (YCbCr model was originally designed to transmit TV signals.)

Each layer is cut into parts of size  $8 \times 8$  pixels and each part is processed separately.

<sup>6</sup> Other linear algebraic concepts are also used for lossy compression, e.g. SVD decomposition.





Instead of encoding the intensity of 64 individual points the image of the entire piece is composed as a linear combination of 64 discrete harmonic functions above the  $8 \times 8$  grid, which is the so-called <u>discrete cosine Fourier transform</u>



Basis of individual points Basis of functions  $\cos(ix)\cos(jy)$ 

Small coefficients can be neglected, yielding a lossy compression. It has much better compression ratio than any lossless encoding, while the changes in the image could hardly be recognized by humans.

# Searching on Google

Search for webpages by keywords is usually performed in two stages:

- 1. First, find all pages that contain those keywords.
- 2. Then sort this group by "relevance".

The calculation of the importance of pages in Google's search engine called the PageRank. was designed according to the *Kendall–Wei theory* ( $\pm$ 1950).

Here the relevance of a page is directly proportional to the sum of relevances of pages that refer to it.



Imagine that links between *all* web pages are encoded in a matrix M, where rows and columns correspond to single pages, s.t.

$$m_{i,j} = \begin{cases} 1 & \text{if the } i\text{-th page refers on the } j\text{-th} \\ 0 & \text{otherwise} \end{cases}$$

We get the so called *adjacency matrix of the web*.

Relevances correspond to a nonnegative vector  $\mathbf{x}$ , satisfying  $M\mathbf{x} = c\mathbf{x}$  for a suitable positive constant c. Thus  $\mathbf{x}$  is an *eigenvector* of M corresponding to the *eigenvalue* c.

It can be shown that with a minor modification of the matrix M, the eigenvalue c of the largest absolute value will be real and positive. Additionally, the eigenvector x corresponding to such eigenvalue has all its components positive.

Some facts about the PageRank method.

- A draft of the method was designed by Larry Page and Sergey Brin. The project began in 1995, the first implementation has been tested in 1998.
- The vector x is updated continually, so that each component is recalculated once a month.
- The order the hypothetical matrix *M*, i.e. the number of indexable pages is more than 5 billions.
  (Data from 2022 according to <u>worldwidewebsize.com</u>.)
- The matrix *M* is of course very sparse it has 7 links per page in average.
- In practice, the adjacency graph of the web is decomposed into smaller blocks, and even those are not represented by a matrix. The vector *x* is calculated by iterative approximate methods (25–80 iterations) with a fixed value of *c* = 0.85.

### Interference minimization in GSM networks

A part of the wireless communication in <u>GSM networks</u> takes place between mobile phones and the co called BTS stations. A station is usually equipped with one or three antennas that serve the neighborhood of the station, called a cell. For the communication within the cell one or more frequencies are used, depending on the number of active GSM phones in the cell. On a single frequency, 6–8 phones can be served.



Frequencies shall be allocated to transmitters so that no or only minimal interference appears. First, frequencies allocated to the transmitters of the same BTS should be treated. One shall also consider interference between the same or similar frequencies of close BTS's. The level of interference may depend on the surrounding terrain and other factors. A suitable plan for the frequency allocation can be obtained by integer linear programming methods.

Binary variables  $x_f^v$  encode whether a transmitter v will be assigned frequency f. Further binary variables  $x_{f,f'}^{v,v'}$  which control the interference between two conflicting frequencies f and f' at close transmitters v and v'.



The conditions of the ILP instance Ax = b guarantee that:

- each transmitter has available an appropriate number of frequencies
- variables x<sup>v,v'</sup><sub>f,f'</sub> correctly determine if an interference appears.

The objective function  $\mathbf{c}^T \mathbf{x}$  is designed to minimize the number of positively evaluated variables  $x_{f,f'}^{\mathbf{v},\mathbf{v}'}$ , i.e. to minimize the total interference.

Realistic instances published on FAP web are of scale

- ► 40-80 frequences for 250-400 transmitters, i.e. 13000-300000 variables x<sup>V</sup><sub>f</sub>
- > 20000-1000000 conditions, i.e. variables  $x_{f,f'}^{v,v'}$

Frequency plans were compiled for some scenarios where the interference dropped by 80-95 % w.r.t. to the plans used.

Use one of these frequency plans led to the reduction of the overhead for the total handover of calls between cells by 20 %.



### Sources of images and further reading

 Stigler's diet problem, finite element method, Jpeg, discrete cosine Fourier transform, Pagerank Wikipedia, <u>Mathworld</u>

 Google search engine structure
 <u>Brin, Page</u>: The anatomy of a large-scale hypertextual web search engine, Proc. WWW 7 (1998) 107–117

 Frequency allocation in GSM networks <u>Eisenblätter</u>: Assigning frequencies in GSM networks Oper. Research Proc. 2002, Springer (2003) 33–40 <u>Eisenblätter et al.</u>: Frequency planning and ramifications of coloring, Disc. Math., Graph Th. 22(2002) 51–88