1. (a) Define the inverse matrix.
   Find matrices $A, B$ such that $AB = I_n$ but $A \neq B^{-1}$.

(b) Define a transposition.
   Determine the sign of a permutation $(5, 3, 4, 1, 7, 6, 2)$ with the help of transpositions.

(c) Define the vector of coordinates.
   In the space of real polynomials of degree at most four determine the vector of coordinates $[f]_X$ of the vector $f(x) = 3x^4 + 3x^3 + x + 3$ with respect to the basis $X = \{x^4 + x^3, x^3 + x^2, x^2 + x, x + 1, x^4 + 1\}$.

2. State and prove a theorem about the relationship between solutions of $Ax = b$ and $Ax = 0$.

3. Write a summary about vector spaces and their subspaces.
   (Please provide definitions, theorem statements, examples and relationships. Proofs are not required.)

(a) $B$ is the inverse matrix of a square matrix $A$ if $AB = I_n$.
   Example: $A$ cannot be square ..., $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $AB = I_2$.

(b) A transposition is a permutation with a single cycle of length 2 and a 1-cycle of length 1.
   The problem: the permutation consists of a single cycle $(1, 5, 2, 3, 4)$ and a fixed point (6). The cycle decomposes as:
   $$(1, 5) \circ (1, 3) \circ (1, 2) \circ (1, 4) \circ (1, 5).$$
   The number of transpositions is 5. The sign of the permutation is $-1$.

Sign of $\pi = -1 \circ -1 = -1$.
c) If $X$ is a finite basis of $V$, $X=(v_1,v_2,\ldots,v_n)$ then the vector of coordinates of $u$ w.r.t. $X$ is $[u]_X= (a_1,\ldots,a_n)^T \in \mathbb{K}^n$ where $u= \sum_{i=1}^{n} a_i v_i$.

The problem: solve $3x^4+3x^3+x+3 = a_1(x^4+x^3+1)+a_2(x^3+x^2+1)+a_3(x^2+x+1)+a_4(x+1)+a_5(x^3+1)$

This leads to a system with matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 3 \\
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 3
\end{pmatrix}^2 \sim \begin{pmatrix}
1 & 0 & 0 & 1 & 3 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \Rightarrow \begin{pmatrix}
a_1 = 2 \\
a_2 = 1 \\
a_3 = -1 \\
a_4 = 2 \\
a_5 = 1
\end{pmatrix} \Rightarrow [u]_X = \begin{pmatrix}
2 \\
1 \\
-1 \\
2 \\
1
\end{pmatrix}
$$

2. If the system $Ax=0$ has at least one solution $x_0$ then the map $f: X \rightarrow X+x_0$ is a bijection between the sets of solutions of $Ax=0$ and $Ax=b$.

Proof: a) $x$ is a solution of $Ax=0 \Rightarrow x+x_0$ is a solution of $Ax=b$:

$A(f(x)) = A(x+x_0) = Ax + Ax_0 = 0 + b = b$.

b) $f(x)$ is a solution of $Ax=b \Rightarrow x$ is a solution of $Ax=0$.

$Ax = Ax + Ax_0 - b = A(x+x_0) - b = Af(x) - b = b - b = 0$.

c) $x \neq x' \Rightarrow f(x) = x+x_0 \neq x'+x_0 = f(x') \Rightarrow f$ is injective.

d) $Ax = b \Rightarrow f^{-1}(x) = x - x_0 \in \ker(A)$ since

$A f^{-1}(x) = A(x-x_0) = Ax - Ax_0 = b - b = 0 \Rightarrow f$ is surjective.
3. Definition: A vector space over a field \( K \) is \((V, +, 0)\) where \((V, +)\) is an Abelian group, \( \cdot : K \times V \rightarrow V \) satisfying:

1) \( \forall u \in V, \lambda \cdot u = u \), where \( 1 \) is the neutral element of \((K, \cdot)\) 
2) \( \forall a, b, c \in K, \forall u \in V, (a \cdot b) \cdot u = a \cdot (b \cdot u) \) 
3) \( \forall a, b \in K, \forall u \in V, (a + b) \cdot u = a \cdot u + b \cdot u \) 
4) \( \forall a \in K, \forall u \in V, a \cdot (u + v) = a \cdot u + a \cdot v \).

Examples: Arithmetic vector space \( K^n \)
- ordered n-tuples of elements of \( K \), \( +, 0 \) coordinate-wise
Analogously (real) sequences / functions on the same domain
in particular polynomials, continuous functions.
All subsets of a set as a vector space over \( \mathbb{Z}_2 \)
+ ... symmetric difference
The set of even subgraphs of a fixed graph
+ ... --, over \( \mathbb{Z}_2 \).

Definition: \( U \) is a subspace of \( V \) if
1) \( \forall u, v \in U, u + v \in U \) 
2) \( \forall a \in K, \forall u \in U, a \cdot u \in U \).

Facts: A subspace is also a vector space over the same field.
The zero vector of \( V \) belongs to all its subspaces.
Theorem: Let \( U_i \) for \( i \in I \) be a collection of subspaces of \( V \). Then \( \bigcap_{i \in I} U_i \) is a subspace of \( V \).

Definition: For a set \( X \subseteq V \) let \( L(X) \), the subspace generated by \( X \), be the intersection of all subspaces containing \( X \).

Definition: A vector \( u \) is a linear combination of vectors from \( X \) if \( \exists k, x_1, x_2, \ldots, x_k \in X, c_1, \ldots, c_k \in \mathbb{K} \) s.t. \( u = \sum_{i=1}^{k} c_i x_i \).

Theorem: For any \( V \), any \( X \subseteq V \) it holds that \( L(X) \) consists of all linear combinations of \( X \).

Formally: \( \bigcap \{ U : X \subseteq U \} = \{ u : u = \sum_{i=1}^{k} a_i x_i, x_i \in X, a_i \in \mathbb{K}, k \in \mathbb{N} \} \).

Examples for \( V = \mathbb{R}^3 \), \( u \neq 0 \), \( L(u) \) is the line passing through \( u \) and \( 0 \).

Analogously, for \( 0, u, v \) non-collinear \( L(\{u, v\}) \) is the plane containing \( u, v \) and \( 0 \).

For \( A \in \mathbb{K}^{m \times n} \), \( \text{Ker}(A) = \{ x : Ax = 0 \} \subseteq \mathbb{K}^n \).

\( \text{R}(A) = L(\text{rows of } A) \subseteq \mathbb{K}^n \).

\( \text{C}(A) = L(\text{columns of } A) \subseteq \mathbb{K}^m \).