Lecture notes: Isomorphism testing on cubic graphs

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The aim of these notes is to survey a part of the Luks's paper from 1982 [1] for educational purposes.

Theorem 1. *The existence of an isomorphisms between cubic graphs can be decided in polynomial time.*

Overview:

1. Given two cubic graphs Y, Y' and create several new graphs X such that Y is isomorphic to Y' if and only if at least one of these X has an automorphism which switches one specified edge $e \in E_X$.
2. To describe automorphisms of X we first find an automorphism on a subgraph of X induced by edges close to e. Then we extend the automorphisms to further distance from e step by step.
3. The extension of automorphisms is reducible to a color automorphism problem. For a colored set A and a group G of permutations on A we want to find generators of automorphisms on G which preserve colors.
4. We split G to two subgroups and/or A to smaller G-orbits so that for each reduced problem we can solve in polynomial time. From the particular solutions we compute a solution for G-orbits and then we compose it to a solution for G.

1 Reduction of isomorphism to automorphism

Assume first that Y and Y' are isomorphic and ψ is such an isomorphism. Choose an arbitrary vertex v of Y and construct a graph X from the disjoint union of Y and Y' by joining v and $v' = \psi(v)$ with a new edge e.

The isomorphism ψ straightforwardly yields an automorphism φ of X that transposes the edge e. If Y and Y' are cubic and we aim to get a cubic X as well we may alter the above by choosing some edge f of Y, and subdivide it by a new vertex v as well as subdivide $\psi(f)$ by v', see Fig. 1.

On the other hand if the two graphs are not isomorphic, then there is no automorphism that transpose the edge e in any possible placement of the vertex v as the subdivision of an edge of Y'.

We reduce a single instance of the graph isomorphism problem to $O(|E_Y|)$ instances of the graph automorphism problem, which for a given graph returns a representation of its automorphism group. A pseudocode of such reduction is described by Algorithm 1.

Note that for bounded degree graphs we have linear dependence between $|V_Y|$ and $|E_Y|$, namely $|E_Y| = \frac{3|V_Y|}{2}$ for cubic graphs.

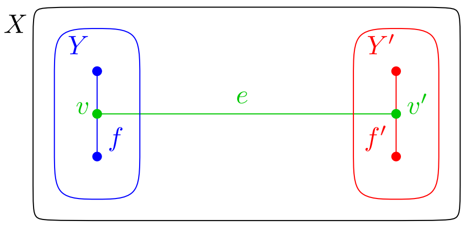
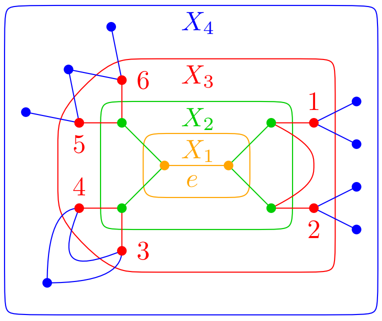


Figure 1: Reduction of graph isomorphism to graph automorphism

Algorithm 1: Reduction of graph isomorphism to graph automorphism

```
   Input  : Graphs $Y$ and $Y'$
   Query  : Is $Y$ isomorphic to $Y'$?
 1 begin
 2     choose an arbitrary edge $f \in Y$;
 3     for each edge $f' \in Y'$ do
 4         $X := Y \cup Y'$;
 5         subdivide in $X$ the edges $f, f'$ with new vertices $v, v'$;
 6         add to $X$ a new edge $e$ between vertices $v, v'$;
 7         if there is an automorphism $\varphi$ of $X$ s.t. $\varphi(v) = v'$ and $\varphi(v') = v$ then
 8             return "$Y$ and $Y'$ are isomorphic";
 9         end
10     end
11     return "$Y$ and $Y'$ are not isomorphic";
12 end
```

a)



b)

$B_e = \{\{3,4\}\}$

$B_1 = \{\{5\},\{6\},\{3,4\},\{5,6\}\}$

$B_2 = \{\{1\},\{2\}\}$

Figure 2: a) The nested subgraphs $X_1, \ldots$ of a graph yield a decomposition into layers X_1, $X_2 \setminus X_1, \ldots$. Notice that the edge $(1,2)$ between two vertices of layer X_1 is contained in X_2.
If we consider only the subgraph X_1 then there is an automorphism which switches the edge e. When we add vertices belonging to the layer $X_2 \setminus X_1$, no such automorphism exists furthermore.
b) the three sets B_e, B_1 and B_2 for $k = 3$.

2 From automorphism extension to colored subgroups

2.1 Decomposition into layers

For a graph X and en edge $e \in E_X$, let the symbol X_k denotes the subgraph of X induced by the edges *of the paths of length at most k that contain e*. Notice that for $k \geq 2$, any edge between two vertices from the layer X_k is contained in the layer X_{k+1}, see Fig. 2 a).

Let the symbol $\mathrm{Aut}_e(X_k)$ be the set of automorphisms φ of X_k that fix e. These namely satisfy either $\varphi(u) = u$ & $\varphi(v) = v$ or $\varphi(u) = v$ & $\varphi(v) = u$, where u, v are the vertices of e.

Finally, let π_k be the projection of the automorphisms of $\mathrm{Aut}_e(X_{k+1})$ onto $\mathrm{Aut}_e(X_k)$, namely $\pi_k(\varphi)$ is the restriction of the automorphism φ to the set of vertices in X_k.

We aim to construct the set of generators of $\mathrm{Aut}_e(X_{k+1})$ as the union of two sets:

- R containing the generators of the kernel of π_k — notice that these automorphisms only switch vertices from the layer $X_{k+1} \setminus X_k$,
- S obtained from the set S' of generators of $\pi_k(\mathrm{Aut}_e(X_{k+1}))$ such that for each $\psi' \in S'$ we insert into S any ψ such that $\pi_k(\psi) = \psi'$. In other words, as each generator of $\pi_k(\mathrm{Aut}_e(X_{k+1}))$ is only an automorphism of X_k, we insert into S any of its extension onto X_{k+1}.

It holds that $\mathrm{Aut}_e(X_{k+1}) = \langle R \cup S \rangle$, as for every $\varphi \in \mathrm{Aut}_e(X_{k+1})$ we may project φ into $\mathrm{Aut}_e(X_k)$, express $\pi_k(\varphi)$ with respect to the generators of $\mathrm{Aut}_e(X_k)$, say $\pi_k(\varphi) = \psi'_1 \ldots \psi'_t$. The corresponding ex-

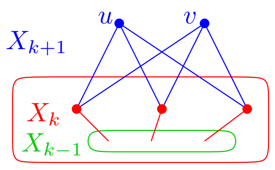


Figure 3: Two vertices u, v in the layer $X_{k+1} \setminus X_k$ are twins. The automorphism of X_{k+1} which switches them, while X_k is fixed is an element of R.

pression with their extensions from S, namely $\psi_1 \dots \psi_t$, agrees with φ on X_k. The difference $\psi_1 \dots \psi_t \varphi^{-1}$ belongs to the kernel of π_k and hence it could be expressed in terms of R.

It is easy to construct the set R, while the difficulties are with the set S.

2.2 Construction of R

We say that vertices $u, v \in X_{k+1} \setminus X_k$ are *twins* if $N(u) \cap X_k = N(v) \cap X_k$, see Fig. 3.

Since the degree of each vertex is 3, each vertex in X_k has at most two neighbors (possible twins) in $X_{k+1} \setminus X_k$ as it must at least one neighbor in X_{k-1} (this holds for all but the two adjacent vertices of X_1).

Therefore, the kernel of π_k is generated by the set of transposition of twins, i.e.

$$R = \{(u, v) \colon u, v \text{ are twins and } u, v \in X_{k+1} \setminus X_k\}$$

2.3 Construction of S

We denote by A the set of all singletons, pairs and triples from $X_k \setminus X_{k-1}$.

$$A = \binom{X_k \setminus X_{k-1}}{3} \cup \binom{X_k \setminus X_{k-1}}{2} \cup \binom{X_k \setminus X_{k-1}}{1}$$

Each automorphism $\varphi \in \mathrm{Aut}_e(X_k)$ corresponds to a permutation $\varphi' \in \mathrm{Sym}(A)$ of elements from A. Consequently the group $\mathrm{Aut}_e(X_k)$ corresponds to some group $G \subseteq \mathrm{Sym}(A)$. To capture the property that φ can be extended into an automorphism of X_{k+1} we introduce three sets A as follows, see Fig. 2 b) for an example:

- $B_e = \{(u, v) \in X_k \setminus X_{k-1} \colon (u, v) \in E(X_{k+1})\}$

  ... these are the new edges inside X_k introduced in the graph X_{k+1},

- $B_1 = \{a \in A \colon \exists! w \in X_{k+1} \setminus X_k \colon N(w) = a\}$

  ... for the tuples that form the neighborhood of one vertex from the new layer $X_{k+1} \setminus X_k$,

- $B_2 = \{a \in A \colon \exists! w, w' \in X_{k+1} \setminus X_k \colon N(w) = N(w') = a\}$

  ... neighborhoods of twins from the new layer.

Observe that B_1 and B_2 are disjoint by the definition as well as B_2 and B_e due to maximum degree 3. Thus an element of A may fall to one of the following subsets which we will view as "colors": B_2, $B_1 \cap B_e$, $B_1 \setminus B_e$, $B_e \setminus B_1$, and $A \setminus (B_e \cup B_1 \cup B_2)$

Lemma 2. *A automorphism φ of X_k belongs to the image $\pi_k(\mathrm{Aut}_e(X_{k+1}))$ if and only if φ' belongs to the subgroup of G which respects the three sets B_e, B_1 and B_2.*

Proof. For the forward implication, if $a \in A$ and $\varphi'(a)$ have different colors then φ can not be an automorphism of X_{k+1}.

For the opposite implication consider an automorphism $\varphi \in \mathrm{Aut}_e(X_k)$. For a vertex $u \in X_{k+1} \setminus X_k$ we denote its neighborhood in the previous layer $N_{X_k}(u)$ by $a_u \in A$.

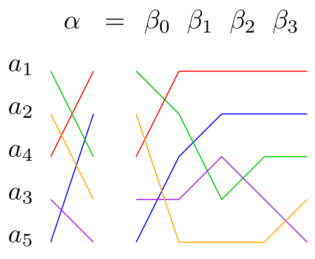


Figure 4: Decomposing α on $n = 5$ elements as a composition of coset representatives $\beta_0 \cdots \beta_{n-2}$.

If $(u, v) \in B_e$ then $(\varphi(u), \varphi(v)) \in B_e$, so φ respects edges newly added in the layer $X_k \setminus X_{k-1}$.

If $a_u \in B_1$ then there exists exactly one u' such that $a_{u'} = \varphi'(a_u)$. We We extend φ on u by setting $\varphi(u) = u'$.

If $a_u \in B_2$ then it has a unique twin $v \in X_{k+1}$ such that $a_v = a_u$. With help of φ' we identify a pair of twins u' and v' such that $a'_v = a'_u = \varphi'(a_u)$. We extend φ on u and v by setting $\varphi(u) = u'$ and $\varphi(v) = v'$. Though there are two possibilities, we choose any of them, since the other could be obtained by the transposition $(u, v) \in R$. $\square$

So the construction of $\mathrm{Aut}_e(X_{k+1})$ now reduces to the problem of how to construct for given generators of $Aut_e(X_k)$ the set of generators of its subgroup that respects some coloring constraints.

3 Representations of permutation groups

The goal of this section is to develop methods that would allow us to represent possibly exponentially large permutation groups and their cosets by polynomially many elements, as well as be able to perform membership tests and unions.

We assume the permutation group $\mathrm{Sym}(A)$ acting on a set A with the composition operation defined for $\alpha, \beta \in \mathrm{Sym}(A)$ as follows: $\forall a \in A : (\alpha\beta)(a) = \beta(\alpha(a))$ where the left to right order of permutations on the composition corresponds to the order of application of them on A. We adopt the notation that Latin letters like a, b or a_i stand for the elements of the ground set A, while Greek minuscules represents group elements, i.e. the permutations.

For a group $G \subseteq \mathrm{Sym}(A)$ and an arbitrary order of the elements of $A = \{a_1, \ldots a_n\}$, we denote by G_i the subgroup of G which fixes each of $a_1, \ldots, a_i$.

Immediately, we get a chain of subgroups:

$$\{id\} = G_n = G_{n-1} \subseteq G_{n-2} \subseteq \ldots \subseteq G_2 \subseteq G_1 \subseteq G_0 = G$$

Definition. *Let G be a group, H be a subgroup and β be any element of G. Then $\beta H = \{\beta\alpha,\ \alpha \in H\}$ is the left* coset *of H in the group G.*

We aim to represent G with sets C_i for $i \in \{0, \ldots, n-2\}$, where each such C_i contains representatives of left cosets of the subgroup G_{i+1} in the group G_i. In other words G_i is the union of left cosets of G_{i+1}, formally written as: $G_i = \bigcup\limits_{\beta \in C_i} \beta G_{i+1}$. Immediately we get that $|G| = \prod\limits_{i=0}^{n-2} |C_i|$.

Our goal is to represent any $\alpha \in G$ by a chain of permutations $\beta_0\beta_1 \cdots \beta_{n-2}$, where each β_i is a coset representative from C_i. As C_i contains only permutations in which the elements a_j for $j < i$ are fixed, the chain $\beta_0\beta_1 \cdots \beta_{n-2}$ can be viewed as a sequence of permutations where one-by-one β_i selects the correct *preimage* of a_{i+1} when $\beta_0, \ldots, \beta_{i-1}$ are already fixed, see Fig. 4.

For this purpose we use the following filtering procedure which for a (not necessarily complete set) of coset representatives either finds such a decomposition of a given α, or extends some of the sets so that it exists.

Observe that the test $\alpha^{-1}(a_{i+1}) = \beta_i^{-1}(a_{i+1})$ is equivalent to $\beta_i^{-1}\alpha \in G_{i+1}$, which could be rephrased that α belongs to the coset of G_{i+1} in G_i represented by β_i. So if no β_i represents such coset, it is appropriate to add at line 8 the present value of α as a new representative.

```
Algorithm 2: Filter($\alpha$)
  Input  : $\alpha \in \mathrm{Sym}(A)$
  Output: $\beta_0, \beta_1, \ldots, \beta_{n-2}$ so that $\beta_i \in C_i$ and $\alpha = \beta_0\beta_1 \cdots \beta_{n-2}$
  Data: sets $C_0, \ldots, C_{n-2}$, where $C_i$ contains (some) coset representatives of $G_{i+1}$ in $G_i$
  begin
      for $i = 0$ to $n-2$ do
          if $\alpha^{-1}(a_{i+1}) = \beta_i^{-1}(a_{i+1})$ for some $\beta_i \in C_i$ then
              $\alpha := \beta_i^{-1}\alpha$;
          end
          else
              $\beta_i := \alpha$;
              add $\beta_i$ to $C_i$;
              $\alpha :=$ id;
          end
      end
      return $\beta_0, \beta_1, \ldots, \beta_{n-2}$
  end
```

Moreover, for a representative β_i the only relevant information was the value of $\beta_i^{-1}(a_{i+1})$. By keeping only one representative for each coset, i.e. representatives that differ on $\beta_i^{-1}(a_{i+1})$, the size of each C_i is at most $n-i$.

In the next step we show how to assure that each element of G has a unique expression with respect to the coset representatives.

Definition. *We say that $C_0, \ldots C_{n-2}$ are the sets of* strong generators *iff*

$$\forall \alpha \in G\ \exists! \beta_0, \beta_1, \ldots, \beta_{n-2} \colon \beta_i \in C_i\ \wedge\ \alpha = \beta_0 \cdots \beta_{n-2}.$$

Lemma 3. *Let $C_0, \ldots C_{n-2}$ be the sets of coset representatives for a chain of subgroups of G. If for each $i, j \in \{0, \ldots n-2\}$ with $i \leq j$ and each $\sigma \in C_i$ and $\tau \in C_j$ holds that $\tau\sigma$ could be expressed as $\beta_0 \cdots \beta_{n-2}$ with $\beta_i \in C_i$ then any $\alpha \in \langle C_0 \cup \cdots \cup C_{n-2}\rangle$ has such expression as well.*

Proof. Assume $\alpha = \gamma_1 \cdots \gamma_t$ and for $j \in \{1, \ldots, t\}$ choose i_j so that $\gamma_j \in C_{i_j}$.

We first determine the set $\{l \colon i_{l-1} \geq i_l\}$. If it is empty then the sequence $\gamma_1 \cdots \gamma_t$ (perhaps padded with the identity maps when necessary) is the desired expression of α.

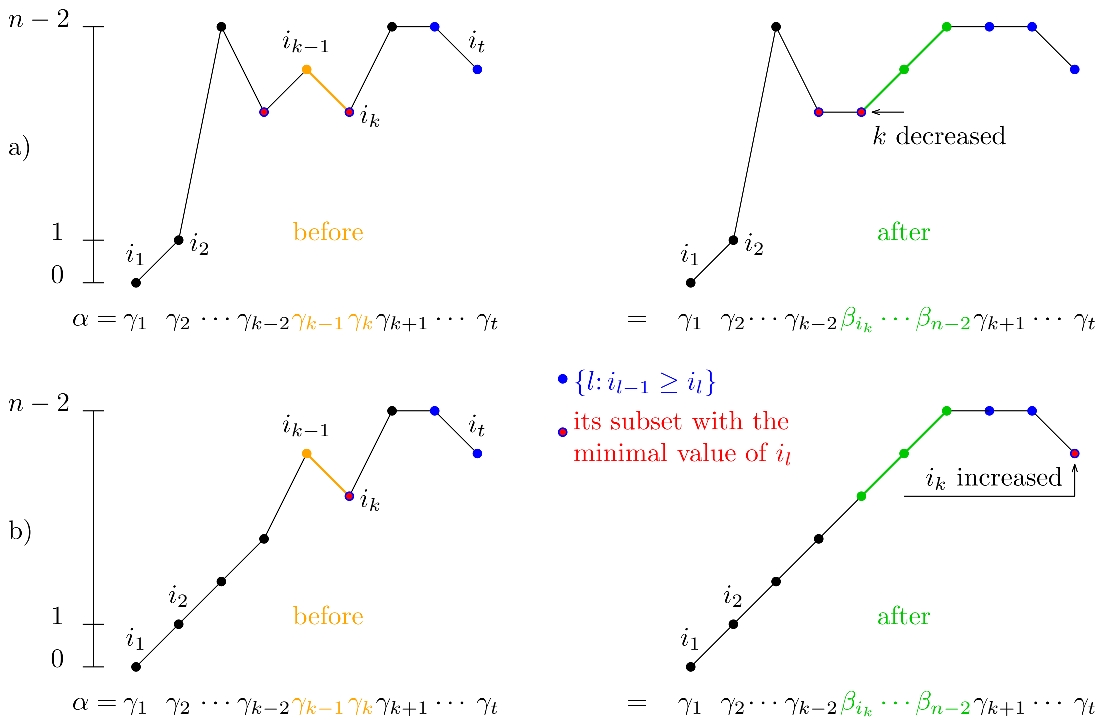
The we select the subset with the smallest value of i_l, and finally, out of these let k be the maximum of this set, see Fig. 5. By the choice, this k has the following property:

- $i_{k-1} \geq i_k$
- $\forall l > k \colon i_l > i_k$
- $\forall i_l < i_k \colon i_{l-1} < i_l$

We apply the lemma assumption for $\sigma = \gamma_k$ and $\tau = \gamma_{k-1}$. Thus $\gamma_{k-1}\gamma_k = \tau\sigma = \beta_{i_k} \cdots \beta_{n-2}$ and if we substitute this term in the expression of α, we obtain $\alpha = \gamma_1 \cdots \gamma_{k-2}\beta_{i_k} \cdots \beta_{n-2}\gamma_{k+1} \cdots \gamma_t$.

Observe that after this replacement each i_l with $l \geq k$ is strictly greater than the former value of i_k. Therefore either the maximum of its subset attaining the minimal value i_l, i.e. the value of k has decreased, see Fig. 5 a) or the minimum of $\{l \colon i_{l-1} \geq i_l\}$ has increased, see Fig. 5 b).

As each index i_l is bounded by $n-2$ and k is non-negative, after finitely many iterations of the above argument (in each round we express the same α but as a different sequence of γ's) we obtain a sequence where the set $\{l \colon i_{l-1} \geq i_l\}$ is empty as wanted. □



$n-2$
a)
1
0
i_1
i_2
i_{k-1}
i_k
i_t
before
$\alpha = \gamma_1 \ \gamma_2 \cdots \gamma_{k-2}\gamma_{k-1}\gamma_k\gamma_{k+1}\cdots \gamma_t$
i_1
i_2
k decreased
after
$= \gamma_1 \ \gamma_2\cdots\gamma_{k-2}\beta_{i_k}\cdots\beta_{n-2}\gamma_{k+1}\cdots\gamma_t$
$n-2$
b)
1
0
i_1
i_2
i_{k-1}
i_k
i_t
before
$\alpha = \gamma_1 \ \gamma_2 \cdots \gamma_{k-2}\gamma_{k-1}\gamma_k\gamma_{k+1}\cdots \gamma_t$
● $\{l\colon i_{l-1} \geq i_l\}$
● its subset with the minimal value of i_l
i_1
i_2
i_k increased
after
$= \gamma_1 \ \gamma_2\cdots\gamma_{k-2}\beta_{i_k}\cdots\beta_{n-2}\gamma_{k+1}\cdots\gamma_t$

Figure 5: Example of the choice of k and the corresponding i_k.
a) The replacement of $\gamma_{k-1}\gamma_k$ with $\beta_{i_k}\cdots\beta_{n-2}$ decreased k. b) The replacement increased i_k.

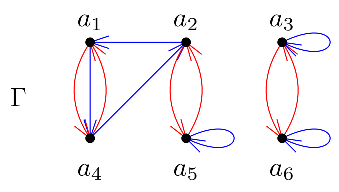
Now we show how a set S generating a group G can be transformed to a set of strong generators. The number of coset representatives might be larger than the size of S. For instance consider the cyclic subgroup of S_5 generated by the cyclic permutation $(2,3,4,5,1)$. This solely permutation generates the cyclic subgroup, but each of its five elements form a unique coset of the (only) subgroup $\{\mathrm{id}\}$, so $|C_1| = 5 > 1 = S = \{(2,3,4,5,1)\}$.

To obtain a set of strong generators we shall not only filter the generators of the subgroup to get coset representatives but also all their possible compositions of to assure that assumptions of Lemma 3 are satisfied. A naïve approach is summarized in the Algorithm 3. We could get more efficient code if a queue of the newly added elements is used instead the repeat–until loop.

Algorithm 3: Strong generators

```
   Input  : A generating set $S$ of a group $G$
   Output: Sets of strong generators $C_0, \dots, C_{n-2}$ for $G$
1  begin
2      for $i := 0$ to $n-2$ do $C_i := \{\mathrm{id}\}$;
3      foreach $\alpha \in S$ do Filter($\alpha$);
4      for $i := 0$ to $n-2$ do
5          repeat
6              foreach $\sigma \in C_i$ do
7                  foreach $\tau \in C_j, j \geq i$ do
8                      Filter($\tau\sigma$)
9                  end
10             end
11         until no coset representative was added by filtration at line 8;
12     end
13 end
```

The correctness of the algorithms follows from the fact that all elements of S were filtered, so $G = \langle C_0 \cup \cdots \cup C_{n-2} \rangle$ and by Lemma 3.



Generators of G (i.e. the set S) in the cyclic notation:

$\alpha : (a_1, a_4, a_2)$

$\beta : (a_1, a_4), (a_2, a_5), (a_3, a_6)$

Figure 6: Example of the graph Γ constructed by Algorithm 4. Here $A = \{a_1, \ldots, a_6\}$, the group G is represented by two generators α (blue) and β (red). The group G has two orbits: $\{a_1, a_2, a_4, a_5\}$ and $\{a_3, a_6\}$. The orbit of a_1 contains a_5, because for $\gamma = \alpha^2\beta \in G$ we have $\gamma(a_1) = a_5$.

The *index* of a subgroup H in a group G is the ratio $\frac{|G|}{|H|}$.

Lemma 4. *Let H be a subgroup of $G \subseteq \mathrm{Sym}(A)$. If H has a polynomial index in G and the membership test for H can be performed in polynomial time then from any set S of polynomially many generators of G we can construct the set of strong generators for H in polynomial time.*

Proof. We alter slightly Algorithm 3 for finding strong generators by adding the set C_{-1} of coset representatives of H in G. This can be done by adding one more filtration step to the beginning of the Algorithm 2. We start with filtration whether $\beta^{-1}\alpha \in H$ for some $\beta \in C_{-1}$ and if not, then we add α to C_{-1}.

From assumptions we know that the additional filtration step can be done in polynomial time. Also because H has a polynomial index in G we add some α to C_{-1} at most polynomially many times.

The resulting set of strong generators for H is $C_0, \ldots C_{n-2}$. □

4 Concepts from group theory

We say that G *acts* on A, or that G is an *action* on A, if $G \subseteq \mathrm{Sym}(A)$. (More properly, an action of a group G on a set A is a homomorphism $G \to \mathrm{Sym}(A)$, but we keep our actions faithful, i.e. injective, and such could be seen just as subgroups of $\mathrm{Sym}(A)$.)

We say that $G \subseteq \mathrm{Sym}(A)$ *stabilizes* $B \subseteq A$, or equivalently that B is G-*stable*, if

$$\forall \alpha \in G \colon \alpha(B) = B.$$

The G-*orbit* of an element $a \in A$ is the set $\{\alpha(a) \colon \alpha \in G\}$. A group G is a *transitive* action, or equivalently that G acts *transitively*, if it has exactly one orbit.

The following algorithm 4 splits a group G into orbits. See Fig. 6 for an example. We later use it for branching to smaller problems by divide & conquer technique.

Algorithm 4: Orbits

```
Input  : Action $G$ on a set $A$ given by a set $S$ of generators, i.e. $G = \langle S \rangle$
Output: Orbits of $G$ on $A$
begin
    create the empty graph $\Gamma$ on the vertex set $A$ ;
    foreach $\alpha \in S$,  $a \in A$ do
        add the edge $(a, \alpha(a))$;
    end
    return connected components of $\Gamma$;
end
```

The *order* of an element α is the smallest $k \in \mathbb{N}$ such that $\alpha^k = \mathrm{id}$. We call a group G a p-*group*, where p is a prime, if each element of G has order p^i for some i. Consequently, the order of G is also a power of p.

Since each subgroup of a p-group is also a p-group [2], then by the construction of sets R and S from subsections 2.2 and 2.3 we have:

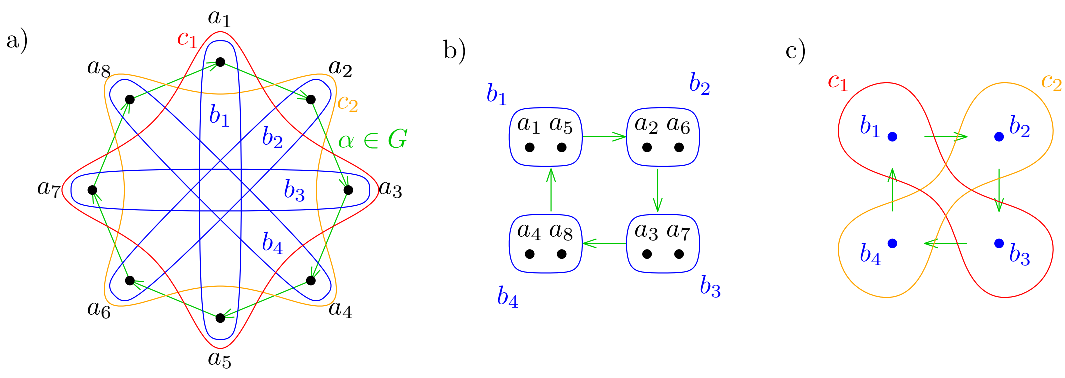


Figure 7: a) Example of two block systems for $G = \langle\alpha\rangle$ on $A = \{a_1, \dots, a_8\}$, where $\{b_1, \dots, b_4\}$ is not minimal and $\{c_1, c_2\} = \{\{a_1, a_3, a_5, a_7\}, \{a_2, a_4, a_6, a_8\}\}$ is minimal. b) The action of α on the blocks $b_1, \dots, b_4$. c) The block system obtained in the second iteration of Algorithm 5 corresponds to the minimal block system $\{c_1, c_2\}$ on A.

Fact 5. *For each k, the group $\mathrm{Aut}_e(X_k)$ is a 2-group.*

We say that a set $B \subseteq A$ is a *G-block* if $\forall\alpha \in G$: $B = \alpha(B)$ or $B \cap \alpha(B) = \emptyset$. If B is a G-block, then the set $\{\alpha(B) \colon \alpha \in G\}$ is a partition of A into disjoint sets of equal size.

Observation 6. *If a group G is transitive on A, then G is transitive also on G-blocks.*

A transitive group G acts *primitively* on A if it does not have blocks of size greater than 1.

Definition. *A nontrivial block system $\mathcal{B} = \{B_1, \dots, B_k\}$, $k \geq 2$ is* minimal *if G acts primitively on its blocks $B_1, \dots, B_k$.*

It means that it has the minimal number of blocks and we cannot merge any remaining blocks, see Fig. 7 a).

The following lemma is well known:

Lemma 7 ([2])**.** *If G is a transitive p-group action on A, $|A| > 1$ then each nontrivial minimal G-block system has exactly p blocks.*

Proof. Let $\mathcal{B}$ be a minimal G-block system. In this proof all actions of G and its subgroups are considered on $\mathcal{B}$.

Denote by G_1 the subgroup of G that stabilizes the first block $B_1 \in \mathcal{B}$, namely $G_1 = \{\alpha \in G : \alpha(B_1) = B_1\}$. Consider a subgroup H of G where $G_1 \subseteq H$.

We claim that $C = \{\sigma B_1 \colon \sigma \in H\}$ is a G-block of $\mathcal{B}$. Whenever $B \in C \cap \alpha(C)$, then $B = \sigma B_1 = \alpha\tau B_1$ for some $\sigma, \tau \in H$. Thus $B_1 = \sigma^{-1}\alpha\tau B_1$, so $\sigma^{-1}\alpha\tau \in G_1$ and therefore $\alpha = \sigma\sigma^{-1}\alpha\tau\tau^{-1} \in H$. Consequently $\alpha(C) = C$, so C is a G-block as it was claimed.

If there was an H strictly between G_1 and G, then $\mathcal{B}$ would not be nontrivial and minimal, which contradicts our assumptions. So G_1 is a maximal subgroup of G. Each maximal subgroup of a p-group has index p. Finally, as G is transitive on $\mathcal{B}$, we get that $|\mathcal{B}| = \frac{G}{G_1} = p$. □

Our goal is to find a minimal block system. We use the following procedure to make from a given block system a new block system with fewer blocks.

Correctness of Algorithm 6 follows from the Lemma 7: When the block system has more than p blocks, there exists i such that a_1, a_i are in the same block of the minimal block system, hence at least graph Γ_i constructed by Algorithm 5 is disconnected. The corresponding block system need not to be minimal, so it is necessary to iterate Algorithm 6 until exactly p blocks are obtained, see Fig. 7 b-c).

Algorithm 5: Block system(A, S)

```
Input  : A set $S$ generating a group action $G$ on $A = \{a_1, \dots, a_n\}$
Output: A $G$-block system on $A$
begin
    for $i = 2$ to $n$ do
        create the empty graph $\Gamma_i$ on the vertex set $A$;
        add $(a_1, a_i) \in E(\Gamma_i)$;
        foreach $(a, a') \in E(\Gamma_i),\ \alpha \in S$ do
            add $(\alpha(a), \alpha(a')) \in E(\Gamma_i)$;
        end
        if $\Gamma_i$ is not connected then
            return connectivity components of $\Gamma_i$;
        end
    end
    return $\{A\}$, because $G$ acts primitively on $A$;
end
```

Algorithm 6: Minimal block system

```
Input  : A set $S$ generating a transitive $p$-group action $G$ on $A$
Output: A minimal $G$-block system on $A$
begin
    $\mathcal{B} = A$;
    repeat
        $\mathcal{B} :=$ Block system$(\mathcal{B}, S)$;
    until $|\mathcal{B}| = p$;
end
```

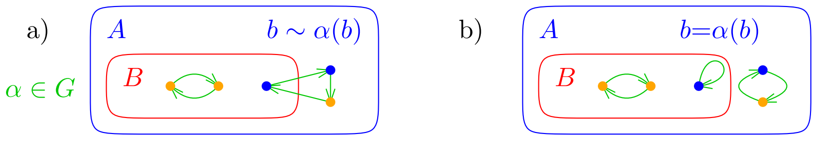


Figure 8: a) Example of a color preserving action α on B. b) A color preserving action α stabilizing B.

5 Finding color preserving subgroups

5.1 Cosets of permutations preserving colors on a block

For elements $a, b \in A$ we write $a \sim b$ when a has the same color as b.

Notation. *For $B \subseteq A$, $K \subseteq \mathrm{Sym}(A)$ we denote by $\mathcal{C}_B(K)$ is the subset of permutations from K preserving colors on elements of B (note that we do not require $\alpha(b) \in B$, see Fig. 8), formally:*

$$\mathcal{C}_B(K) = \{\alpha \in K \colon \forall b \in B \colon b \sim \alpha(b)\}.$$

The goal is to determine $\mathcal{C}_A(G)$. We solve this problem in more general setting, namely, we determine $\mathcal{C}_B(\beta G)$, where B is G-stable, $G \subseteq \mathrm{Sym}(A)$ and $\beta \in \mathrm{Sym}(A)$. Then for $B = A$ and $\beta = \mathrm{id}$ we get $\mathcal{C}_A(G)$.

The two following observations are immediate:

Observation 8. $\mathcal{C}_B(K \cup K') = \mathcal{C}_B(K) \cup \mathcal{C}_B(K')$

Observation 9. $\mathcal{C}_{B \cup B'}(K) = \mathcal{C}_B(\mathcal{C}_{B'}(K))$

Lemma 10. *If B is G-stable then either $\mathcal{C}_B(\beta G) = \emptyset$ or $\mathcal{C}_B(\beta G)$ is a coset of a subgroup $\mathcal{C}_B(G)$ of G.*

Proof. As B is G-stable then $\forall \sigma, \tau \in \mathcal{C}_B(G)$ we have that σ and τ preserve colors on B, namely $\forall b \in B \colon \sigma(b) \sim \tau(b) \sim b$. Consequently, their composition $\sigma\tau$ preserves colors as well, i.e. $\sigma\tau(b) \sim b$ and we get that $\sigma\tau \in \mathcal{C}_B(G)$. Hence $\mathcal{C}_B(G)$ is a subgroup of $\mathrm{Sym}(A)$.

If $\mathcal{C}_B(\beta G) \neq \emptyset$ then there exists $\gamma \in \mathcal{C}_B(\beta G)$. Especially $\gamma \in \beta G$ and so $\beta G = \gamma G$, as we may change the coset representative from β to γ. Because γ is color preserving on B, the following implication holds:

$$\forall \alpha \in G, \forall b \in B : \alpha(b) \in B \Rightarrow \gamma\alpha(b) \sim \alpha(b),$$

The assumption of the implication is valid for all $\alpha \in G$ as B is G-stable, so we obtain:

$$\alpha \in \mathcal{C}_B(G) \Leftrightarrow \gamma\alpha \in \mathcal{C}_B(\gamma G)$$

because either both $\alpha(b)$ and $\gamma\alpha(b)$ have the same color as b or none of them.

Therefore $\mathcal{C}_B(\gamma G) = \gamma\mathcal{C}_B(G)$, in other words $\mathcal{C}_B(\gamma G)$ is a coset of $\mathcal{C}_B(\beta G)$. □

5.2 Finalizing the algorithm

In order to determine $\mathcal{C}_B(\rho G)$ we distinguish two cases:

Case 1. *The group G does not act transitively on B, see Fig. 9 a).*

There is more than one G-orbit on B, so there exist blocks B', B'' such that $B = B' \cup B''$ and both B', B'' are G-stable. We use the Observation 9 and get:

$$\mathcal{C}_B(\beta G) = \mathcal{C}_{B'}(\mathcal{C}_{B''}(\beta G))$$

Case 2. *The group G acts transitively on B, see Fig. 9 b).*

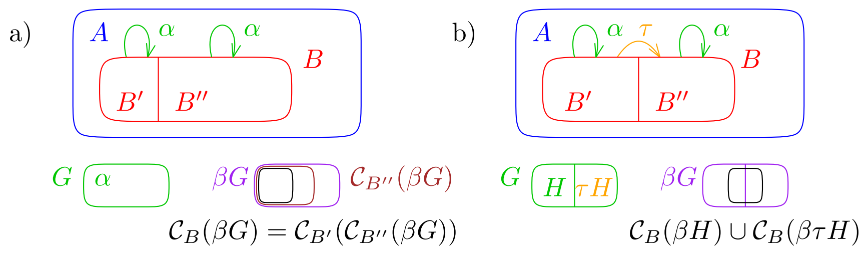

a)
A
α
α
B
B′
B″
G
α
βG
$\mathcal{C}_{B''}(\beta G)$
$\mathcal{C}_B(\beta G) = \mathcal{C}_{B'}(\mathcal{C}_{B''}(\beta G))$
b)
A
α
τ
α
B
B′
B″
G
H
τH
βG
$\mathcal{C}_B(\beta H) \cup \mathcal{C}_B(\beta\tau H)$

Figure 9: a) Case 1 — G does not act transitively on B. b) Case 2 — G acts transitively on B.

Due to Fact 5 and Lemma 7 we use Algorithm 6 and split B into two G-blocks B' and B'' such that $B = B' \cup B''$.

In the next step we determine generators of a subgroup H, which stabilizes B' as follows: The membership of any permutation α in H can be decided in polynomial time by checking whether $\forall b \in B' \colon \alpha(b) \in B'$. Also as $G = H \cup \tau H$, where τ is a permutation which switches blocks B' and B'', we get that the index of H in G is 2. So we can compute generators of the subgroup H from generators of G in polynomial time due to Lemma 4. Note that H stabilizes B'' as well.

Due to Observation 8 (for $G = H \cup \tau H$) and Observation 9 (for $B = B' \cup B''$) the following continued equality holds:

$$\mathcal{C}_B(\beta G) = \mathcal{C}_B(\beta H \cup \beta\tau H) = \mathcal{C}_B(\beta H) \cup \mathcal{C}_B(\beta\tau H) = \mathcal{C}_{B'}\mathcal{C}_{B''}(\beta H) \cup \mathcal{C}_{B'}\mathcal{C}_{B''}(\beta\tau H)$$

By Lemma 10 the first set $\mathcal{C}_{B'}\mathcal{C}_{B''}(\beta H)$ is either empty or could be expressed as $\gamma\mathcal{C}_B(H)$ for a suitable $\gamma \in G$. Similarly, the second set is either empty or could be written as $\delta\mathcal{C}_B(H)$ for a $\delta \in G$.

If both are nonempty then Lemma 10 guarantees that $\mathcal{C}_{B'}\mathcal{C}_{B''}(\beta H)$ and $\mathcal{C}_{B'}\mathcal{C}_{B''}(\beta\tau H)$, or equivalently $\gamma\mathcal{C}_B(H)$ and $\delta\mathcal{C}_B(H)$, "must paste together neatly to a single coset"[1], namely $\gamma\,\langle\mathcal{C}_B(H) \cup \gamma^{-1}\delta\rangle$.

It remains to describe the situation when the recursion stops, i.e. when $|B| = 1$. Then we have directly

$$\mathcal{C}_B(\beta G) = \mathcal{C}_{\{b\}}(\beta G) = \begin{cases} \beta G & \text{if } \beta(b) \sim b, \\ \emptyset & \text{otherwise.} \end{cases}$$

References

[1] Luks, Eugene M. *Isomorphism of graphs of bounded valence can be tested in polynomial time*. Journal of computer and system sciences, 1982.

[2] Wielandt, Helmut. Finite permutation groups. New York: Academic Press, 1964.