A new lower bound for sphere packing

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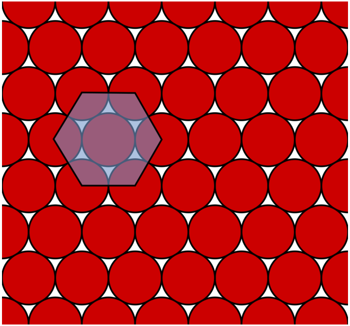
$d = 2$: $\theta(d) \approx 0.90960$ Lagrange (1773)*, Thue (1890), Tóth (1942)

(*for lattice sphere packings*)



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$d = 3$: $\theta(d) \approx 0.74048$ Gauss (1831)*, Hales (1998)

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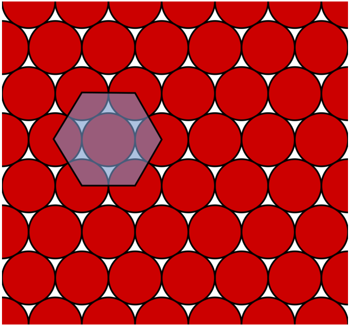

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Proof.

- Consider a maximal sphere packing $\mathcal{P}$
- We can cover the whole $\mathbb{R}^d$ by doubling the radius of each ball in $\mathcal{P}$
- Hence $\theta(d) \geq 1/2^d$, as required □

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Remark: Up to a constant factor, the best known upper bound on $\theta(d)$ is $1/2^{(0.599\ldots\ +\ o(1))d}$ by Kabatjanskii and Levenstein (1978).

Order vs. disorder

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$$\mathcal{L} = \{\alpha_1 x_1 + \ldots + \alpha_d x_d \colon x_1, \ldots, x_d \in \mathbb{Z}\}$$

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- Optimal solution for $d \in \{1, 2, 3, 8, 24\}$ is achieved by a lattice sphere packing
- However, lattice sphere packings are provably not optimal for some dimensions d (e.g. $d = 10$)

Formal definition of Sphere packing problem

- Notation:
  - $B_0(R)$ ... the ball in $\mathbb{R}^d$ of radius R centered in the origin
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- Find in G_X independent set of size $(1 - o(1))\mathrm{Vol}(B_0(R))\frac{d \log d}{2^{d+1}}$

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Poisson point process $\mathbf{X}$ has the following properties:

- *(Poisson distribution of points count)* For a Borel set $B \subseteq B_0(R)$, we have $|\mathbf{X} \cap B| \sim Pois(\lambda \cdot \mathrm{Vol}(B))$
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Recall that $\mathbf{Y} \sim \mathit{Pois}(\lambda)$ denotes a Poisson random variable with intensity λ; that is $\Pr[\mathbf{Y} = k] = \lambda^k e^{-\lambda}/k!$ for $k = 0, 1, 2, \ldots$

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- $\implies \alpha(G_X) \geq \frac{|X|}{\Delta(G_X)+1} \approx \frac{\lambda \cdot \mathrm{Vol}(B_0(R))}{\lambda \cdot 2^d} = \frac{\mathrm{Vol}(B_0(R))}{2^d}$

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- $\implies \theta(d) \geq 1/2^d$

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$$\alpha(G_X) \geq \alpha(G_{X'}) \geq (1 - o(1))\frac{|X'| \log(\Delta(G_{X'}))}{\Delta(G_{X'})} \approx (1 - o(1))\frac{|X| \log(\Delta(G_X))}{\Delta(G_X)}$$

$$\approx (1 - o(1))\frac{|X| \log(\lambda \cdot d)}{\lambda \cdot 2^d} = (1 - o(1)) Vol(B_0(R))\frac{c \cdot d}{2^d}$$

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- $\lambda = (\frac{\sqrt{d}}{8 \log d})^d$
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Lemma (S)

There exists a set of points $X \subseteq B_0(R)$ such that

$$|X| \geq (1 - o(1))\lambda \cdot Vol(B_0(R))$$

and

$$\Delta(G_X) \leq \Delta(1 + \Delta^{-1/3}) \text{ and } \Delta_2(G_X) \leq \Delta(\log \Delta)^{-\log\log \Delta},$$

where $\Delta = \lambda \cdot 2^d = \left(\frac{\sqrt{d}}{4 \log d}\right)^d$.

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Lemma (G)

If $\Delta_2(G) \leq C\Delta(G)/(\log \Delta(G))^c$, then $\alpha(G) \geq (1 - o(1))\frac{|G|\log\Delta(G)}{\Delta(G)}$.

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- By Lemma (S), there exists $X \subseteq B_0(R)$ with $|X| \geq (1 - o(1))\lambda \cdot Vol(B_0(R))$ for which

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- And by Lemma (G), we have

$$\begin{aligned}
\alpha(G) &\geq (1 - o(1)) \frac{|X| \log \Delta(1 + \Delta^{-1/3})}{\Delta(1 + \Delta^{-1/3})} \geq (1 - o(1)) \frac{|X| \log \Delta}{\Delta} \\
&= (1 - o(1)) \mathrm{Vol}(B_0(R)) \cdot \lambda \frac{\log \Delta}{\Delta} \\
&= (1 - o(1)) \mathrm{Vol}(B_0(R)) \frac{1}{2^d} \cdot \log \left(\frac{\sqrt{d}}{4 \log d} \right)^d \\
&\geq (1 - o(1)) \mathrm{Vol}(B_0(R)) \frac{1}{2^d} \cdot \frac{d \log d}{2}.
\end{aligned}$$

$\square$

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Proof of the theorem.

- By Lemma (S), there exists $X \subseteq B_0(R)$ with $|X| \geq (1 - o(1))\lambda \cdot Vol(B_0(R))$ for which

$$\Delta(G_X) \leq \Delta(1 + \Delta^{-1/3}) \ \text{ and } \ \Delta_2(G_X) \leq \Delta(\log \Delta)^{-\log\log\Delta}.$$

- And by Lemma (G), we have

$$\begin{aligned}
\alpha(G) &\geq (1 - o(1))\frac{|X| \log \Delta(1 + \Delta^{-1/3})}{\Delta(1 + \Delta^{-1/3})} \geq (1 - o(1))\frac{|X| \log \Delta}{\Delta} \\
&= (1 - o(1))\mathrm{Vol}(B_0(R)) \cdot \lambda \frac{\log \Delta}{\Delta} \\
&= (1 - o(1))\mathrm{Vol}(B_0(R))\frac{1}{2^d} \cdot \log\left(\frac{\sqrt{d}}{4\log d}\right)^d \\
&\geq (1 - o(1))\mathrm{Vol}(B_0(R))\frac{1}{2^d} \cdot \frac{d \log d}{2}.
\end{aligned}$$

$\square$

Remark: Lemma (G) is tight (up to the constant c and C).

A newer lower bound for sphere packing

Theorem (Klartag; 2025)

$$\theta(d) \geq c \cdot \frac{d^2}{2^{d+1}}$$

for some absolute constant c.

References

[1] Д.Ильин, "Circle packing" (2025), via Wikimedia Commons, Public Domain

[2] Funkdooby, "Cannonballs in the garden at Rye Castle" (2011), via Wikimedia Commons, CC BY 2.0.

Thank you!