The asymptotics of $r(4, t)$

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Ramsey theorem and Ramsey numbers

Theorem

For every $s, t \in \mathbb{N}$, there is a value $r(s,t)$ such that any graph on at least $r(s,t)$ vertices contains either a clique of size s or an independent set of size t.

- The Erdős-Szekeres's proof gives the relation
$$r(s,t) \leq r(s-1,t) + r(s,t-1).$$
- The relation is satisfied by $\binom{r+s-2}{r-1}$.
- Thus, $r(s,s) \leq 4^s$.
- Also, for any fixed $s \in \mathbb{N}$, $r(s,t) \in O(4^{t-1})$.
- Moreover, it is known that for fixed $s \in \mathbb{N}$
$$r(s,t) \in (1+o(1))\frac{t^{s-1}}{(\log t)^{s-2}}$$
by Ajtai, Komlós, Szemerédi.

Main theorem

Theorem

As $t \to \infty$,

$$r(4, t) \in \Omega\left(\frac{t^3}{(\log t)^4}\right).$$

Lower bounds

Task: for given $s, t \in \mathbb{N}$, find a graph without K_s and $\overline{K}_t$ as large as possible.

Common approaches:

- Randomness
- Algebraic construction

Finite projective planes

Finite projective plane is a finite set system $(X, \mathcal{P})$ satisfying:

1. for every distinct $P_1, P_2 \in \mathcal{P}$, there is a unique $x \in X$ such that $x \in P_1 \cap P_2$,
2. for every distinct $x_1, x_2 \in X$, there is a unique $P \in \mathcal{P}$ such that $x_1, x_2 \in P$,
3. there is a set $S \subseteq X, |S| = 4$, with the property that for each $P \in \mathcal{P}$ contains at most two point of S.

Basic properties of FPP

Let $(X, \mathcal{P})$ be a finite projective plane. There is a value $n \in \mathbb{N}$ such that

1. (i) each $x \in X$ lies in exactly $n+1$ lines,
2. (ii) each $P \in \mathcal{P}$ contains exactly $n+1$ points,
3. (iii) $|X| = n^2 + n + 1$,
4. (iv) $|\mathcal{P}| = n^2 + n + 1$.

There is a FPP of each order q that is a power of a prime. We denote this FPP by FPP(q).

Using FPP

$\text{ex}(n, C_4)$ = the maximum number of edges of a C_4-free graph on n vertices

Known: $\text{ex}(n, C_4) \in O(n^{3/2})$ (by counting $K_{1,2}$; cherries)

Proposition

There are arbitrarily large C_4-free graphs on n vertices with $\Omega(n^{3/2})$ edges.

Proof.

Let $(X, \mathcal{P})$ be a projective plane of order p and consider its incidence graph $G = (V, E)$. Note that $C_4 \not\subseteq G$. Moreover,

$$|V| = 2(p^2 + p + 1),$$
$$|E| = (p^2 + p + 1)(p + 1) \geq (p^2 + p + 1)^{3/2}.$$

Thus, $|E| \geq c|V|^{3/2}$ for some fixed $c \in \mathbb{R}$. □

Constructing FPP from a field

Let $\mathbb{F}_n$ be a finite field with n elements. We define $(X, \mathcal{P})$ of order n as follows:

1. We set X to be the lines of $\mathbb{F}_n^3$ (subspaces of dimension 1), i.e. vectors $(x, y, z) \in \mathbb{F}_n^3 \setminus \{(0, 0, 0)\}$ factorized by scalar multiplication,
2. moreover, $\mathcal{P}$ is the set of planes (subspaces of dimension 2) such that each $P \in \mathcal{P}$ is formed by the lines (members of X) that are contained in the plane.

That is, if P is represented by the normal vector $(a, b, c) \in \mathbb{F}_n^3 \setminus \{(0, 0, 0)\}$, then P contains exactly those vectors (x, y, z) for which

$$ax + by + cz = 0.$$

Overview

- Using the magical properties of FPP, we obtain a graph H_q with
  - $n = q^2(q^2 - q + 1)$ vertices,
  - edges are a union of $q^3 + 1$ edge-disjoint cliques of order q^2,
  - each copy of K_4 in H_q has at least three vertices in one of these cliques.
- We consider the random n-vertex graph H_q^* as a union of complete bipartite subgraphs of the cliques of H_q (hence, H_q^* is K_4-free).
- There is an instance G_q^* of H_q^* with at least $2^{40}q^3$ edges; using the container method, it has at most $(q/\log^2 q)^t$ independent sets of size $t = 2^{30} q \log^2 q$.
- Thus, by sampling vertices with probability $(\log^2 q)/q$, we obtain a graph with at least $(q^3 \log^2 q)/2$ vertices and no independent sets of size t, yielding $r(4, t) \geq ct^3/\log^4 t$.

The FPP magic – Hermitian unital

Consider the following subset of points of FPP(q^2)

$$\mathcal{H} = \{(x.y.z) : x^{q+1} + y^{q+1} + z^{q+1} = 0\}.$$

It satisfies the following:

1. $|\mathcal{H}| = q^3 + 1$,
2. every line of FPP(q^2) intersects $\mathcal{H}$ either in 1 (tangents) or $q+1$ points (secants).

Restricting FPP(q^2) to the points of $\mathcal{H}$ and secants $\mathcal{S}$ of $\mathcal{H}$ yields a Steiner $(q+1)$-tuple system.

Moreover, the restriction does not contain the *O'Nan configuration*.

The FPP magic – the graph H_q

We define H_q as follows:

$$V(H_q) = \mathcal{S},$$
$$E(H_q) = \{(S, S') : \text{ there is } x \in \mathcal{H}, \text{ such that } x \in S \cap S'\}.$$

Then,

1. (i) $|V(H_q)| = q^2(q^2 - q + 1)$,
2. (ii) there is a set $\mathcal{C}$ of $q^3 + 1$ maximal cliques of order q^2, every two sharing exactly one vertex,
3. (iii) each vertex lies in exactly $q + 1$ cliques of $\mathcal{C}$,
4. (iv) every copy of K_4 in H_q contains at least three vertices in some clique of $\mathcal{C}$.

Moreover, for each $X \subseteq V(H_q)$ of size $2^{24}q^2$, many edges of $H_q[X]$ lie in many cliques.

The random K_4-free graph H_q^*

For each maximal clique $C \in \mathcal{C}$ of H_q, let (A_C, B_C) be a random partition of $V(C)$ such that each $v \in V(C)$ belongs to A_C, resp. B_C, with probability $1/2$ (independently).

Then H_q^* is the random graph consisting of the union over all maximal cliques $C \in \mathcal{C}$ of the complete bipartite subgraph with parts A_C and B_C.

Theorem

There is a realization of G_q^* of H_q^* such that for every set $X \subseteq V(G_q^*)$ of size at least $2^{24}q^2$,

$$e(G_q^*[X]) \geq \frac{|X|^2}{256q}.$$

The container method

Proposition

Let G be a graph on n vertices, and let $r, R \in \mathbb{N}$, and $\alpha \in [0,1]$ satisfy:

$$e^{-\alpha r} n \leq R,$$

and, for every subset $X \subseteq V(G)$ of at least R, vertices,

$$2e(X) \geq \alpha |X|^2.$$

Then the number of independent sets of size $t \geq r$ in G is at most

$$\binom{n}{r}\binom{R}{t-r}.$$

Randomly sampling from G_q^*

- By applying the container method appropriately, we obtain that G_q^* has at most $(q/\log^2 q)^t$ independent sets of size $t = 2^{30} q \log^2 q$.
- Randomly sample a set V of vertices of G with probability $\log^2 q/q$ independently for each vertex.
- Then the expected number of independent sets of size t is at most 1.
- It follows that there is a K_4-free graph with

$$\frac{q^3 \log^2 q}{2} \geq c \frac{t^3}{\log^4 t}$$

vertices and no independent set of size t.

Proof review

What was key in the proof?

To obtain the graph G_q^* with the properties:

- K_4-free,
- every sufficiently large set induces many edges.

Unsurprisingly.

(n, d, λ)-graphs

We say that G is an (n, d, λ)-graph if

1. $|V(G)| = n$,
2. G is d-regular,
3. the second largest eigenvalue (in the absolute value) is λ.

(n, d, λ)-graphs are pseudorandom

Theorem (Expander mixing lemma)

Let G be an (n, d, λ)-graph and let $X \subseteq V(G)$. Then

$$\left| 2e(X) - \frac{d}{n}|X|^2 \right| \leq \lambda |X|.$$

The smaller value of λ, the better.

The value λ cannot be too small

Theorem (Alon-Boppana)

Let $d \geq 1$. If G is a d-regular graph, then

$$\lambda \geq 2\sqrt{d-1}.$$

In light of this theorem, we refer to an (n, d, λ)-graph with $\lambda \in O(\sqrt{d})$ as a *spectrally extremal graph*.

They exist for arbitrarily large fixed d (Ramanujan graph, 1980's), but also for d depending on n.

K_s-free graph cannot have d too large

Theorem (Sudakov, Szabo and Vu)

Let G be a K_s-free (n, d, λ)-graph. Then

$$d \in O\left(\lambda^{\frac{1}{s-1}} n^{1-\frac{1}{s-1}}\right).$$

If G is spectrally extremal, this gives

$$d \in O\left(n^{1-\frac{1}{2s-3}}\right).$$

The best we know is $d \in \Theta(n^{1-\frac{1}{s-1}})$.

Application of K_s-free (n, d, λ)-graphs

Theorem (Mubayi, Verstaete)

If there exists a spectrally extremal K_s-free (n, d, λ)-graph with $d \in O(n^{1-1/(2s-3)})$, then

$$r(s,t) \in \Omega\left(\frac{t^{s-1}}{(\log t)^{2s-4}}\right).$$

Recall that it is known

$$r(s,t) \leq (1+o(1))\frac{t^{s-1}}{(\log t)^{s-2}}.$$

Counting independent sets again

The previous theorem relies on the container method, which implies:

Proposition

If G is an (n, d, λ)-graph, then the number of independent sets of size $t \geq 2n(\log n)^2/d$ is at most

$$\left(\frac{4e^2\lambda}{\log^2 n}\right)^t.$$

Counting independent sets again – better?

If there is $C > 0$ such that the number of independent sets of size $t \geq C(2n \log n)/d$ in G is at most

$$\left(\frac{C\lambda}{\log n}\right)^t,$$

the previous theorem would give (conditionally) that

$$r(s,t) \in \Omega\left(\frac{t^{s-1}}{(\log t)^{s-2}}\right).$$

Thank you.

Questions?