The Analytic Arc Cover Problem

and its Applications to Contiguous Art Gallery, Polygon Separation, and Shape Carving

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Art Gallery

- Art Gallery = simple polygon P
- Guard = point in P
- Guard x sees a point $y \in P$ = segment $xy \subset P$

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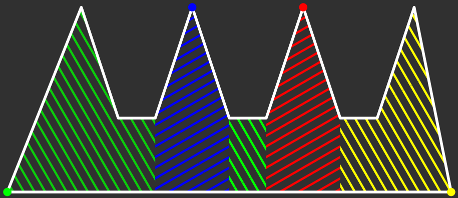
Art Gallery

Question

Given an art gallery P with n vertices, how many guards are needed to guard the whole gallery?

Art Gallery



Art Gallery

Answer:

Theorem (Chvátal)

Any art gallery P can be guarded by $\lfloor \frac{n}{3} \rfloor$ guards, and sometimes this number of guards is necessary.

Art Gallery - computational problems

ArtGallery	
Input	Art gallery P and an integer k.
Question	Is there a set of k guards in P such that every $y \in P$ is visible by some guard.

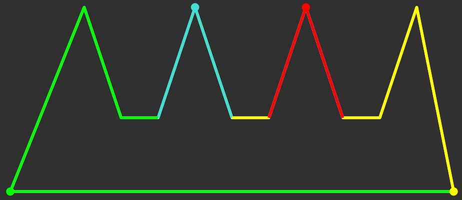
Theorem (Aggarwal 1984)

ArtGallery *is NP-hard*

Art Gallery - computational problems

BoundaryArtGallery	
Input	Art gallery P and an integer k.
Question	Is there a set of k guards in P such that every point on the boundary of P is visible by some guard.

Art Gallery - computational problems



Art Gallery - computational problems

BoundaryArtGallery	
Input	Art gallery P and an integer k.
Question	Is there a set of k guards in P such that every point on the boundary of P is visible by some guard.

Theorem (Lee and Lin 1986)

BoundaryArtGallery *is NP-hard.*

Art Gallery - computational problems

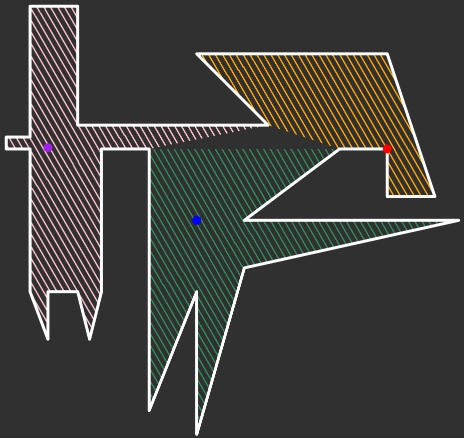
Question (Thomas C. Shermer)

Is the guarding of disjoint regions necessary for the hardness proofs of ArtGallery *and variations like* BoundaryArtGallery*?*

Contiguous Art Gallery problem

	ContiguousArtGallery
Input	Art gallery P and an integer k.
Question	Is there a set of k contiguous paths on boundary of P, covering the entire boundary such that each path is fully visible from some point in P?

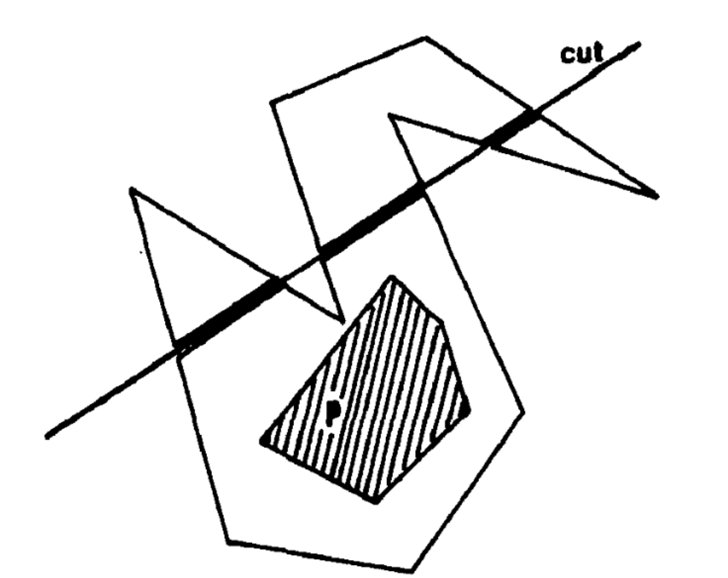
Contiguous Art Gallery problem



Minimum Polygon Separation

POLYGONSEPARATION	
Input	Simple polygon P,a convex polygon Q such that P is contained in Q and a number k.
Question	Is there a polygon S with k vertices such that $P \subset S \subset Q$?

Minimum Polygon Separation



Minimum Polygon Separation

5. **References.**

No references on this topic seem to exist and no useful results could be found.

Minimum Polygon Separation

	POLYGONSEPARATION
Input	Simple polygon P,a convex polygon Q such that P is contained in Q and a number k.
Question	Is there a polygon S with k vertices such that $P \subset S \subset Q$?

Theorem (Aggarwal et al. 1989)

POLYGONSEPARATION *can be solved in* $O(n \log n)$ *time. Moreover, any minimal solution* S *is a convex polygon.*

Segment Separation

SegmentSeparation	
Input	Two sets of line segments A and B in $\mathbf{R}^2$ and a number k.
Question	Is there a convex polygon S on k vertices separating A from B?

Analytic Arc Cover problem

- $\mathcal{I}$ = set of half-open arcs covering S^1
- Next generator for $\mathcal{I}$ = function $g : S^1 \to S^1$ s.t.

$$g(t) = \sup\{b | [a, b) \in I \wedge t \in [a, b)\}$$

Analytic Arc Cover problem

ANALYTICARCCOVER	
Input	Set $\mathcal{I}$ of arcs covering S^1 with next-generator g and an integer k.
Question	Is there an x such that $$[x, g(x)) \cup [g(x), g(g(x))) \cup \cdots \cup [g^{k-1}(x), g^k(x))$$ covers S^1?

Analytic Arc Cover problem

Theorem

If g is a piecewise linear rational function for either the unit-interval or ray representations of S^1, the AnalyticArcCover *problem can be solved in time polynomial in the size of the optimal solution k, the combined bit-complexity of the end points of the pieces and each linear rational function.*

Application to Art Gallery Problem

Theorem

$\textsc{ContiguousArtGallery}$ *problem is in* P*.*

Application to Art Gallery

Idea: Reduce to $\textsc{AnalyticArcCover}$

- Circle is homeomorphic to P
- $\mathcal{I}$ = subpaths of the boundary of P
- Next-generator g is then easily defined.
- We need to prove that g is a linear rational function.

Application to Segment Separation

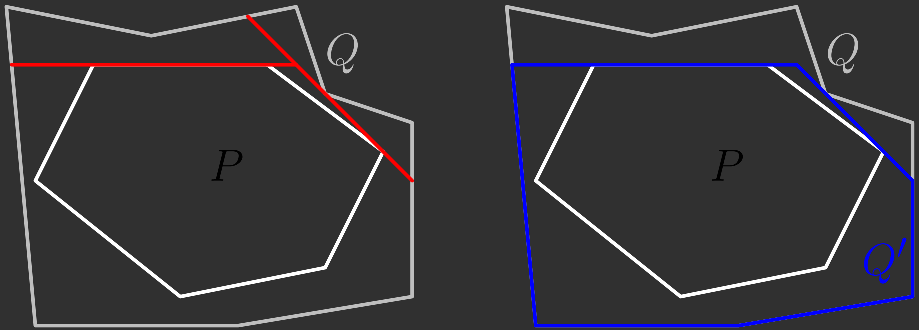
Theorem

SEGMENTSEPARTION *is in P.*

Application to segment separation

Lemma

Let P be a convex polygon contained in another polygon Q, then there is a convex polygon $Q' \subset Q$ with at most as many vertices as Q, containing P.



Application to Segment Separation

Idea:

- Instead of separating segments, separate convex hull of the endpoints of the segments
- map each possible cut to a point on S^1 via its normal vector
- Figure out how to represent next cut
- Make this function a linear rational function

Application to Segment Separation

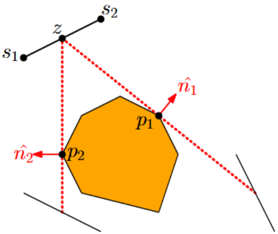


Figure 12 A linear rational function for the next half-plane by its normal vector $\hat{n}_2$ from $\hat{n}_1$, with p_1, p_2, s_1, s_2 all fixed. In particular, $z = s_1 + \frac{\hat{n}_1 \cdot (p_1 - s_1)}{\hat{n}_1 \cdot (s_2 - s_1)}(s_2 - s_1)$ and $\hat{n}_2 = ((p_2 - z)_y, -(p_2 - z)_x)$.

The End