On the Number of Birch Partitions

Author: Stephan Hell
Presented by: Haochi Jiang

May 15

1 Introduction

Definition 1 Let X be a set of $k(d+1)$ points in $\mathbb{R}^d$ for some $k > 1$. A point $p \in R^d$ is a *Birch point* if there is a partition of X into k subsets of size $d+1$, each contains p in its convex hull. The partition of X is a *Birch partition for* p. For fixed $p \in \mathbb{R}^d$, let $B_p(X)$ be the number of unordered Birch partition for p.

Definition 2 A set of points in $\mathbb{R}^d$ is in *general position* if no $k+2$ points are on a common k-dimensional affine subspace. A set X of points in $\mathbb{R}^d$ is in *general position with respect to a point* p if $X \cup \{p\}$ is in general position.

Theorem 3 Let X be a set of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of X into r subsets such that their convex hulls contain a common point. And the partition of X is a *Tverberg Partition.*

2 The Result

Lemma 4 Let X be a set of $d+2$ points in $\mathbb{R}^d$ that is in general position with respect to the origin. Then the number of d-simplices with vertices in X that contain the origin is even. In fact, this number is either 0 or 2.

Lemma 5 Let X be set of $(d+1)(q-1)+1$ in general position in $\mathbb{R}^d$. Then a Tverberg partition consists of:

1. One vertex v and $(q-1)$ $d-$simplices containing v.
2. k intersecting simplices of dimension less than d and $(q-k)$ $d-$simplices containing the intersection point for some $1 < k \leq \min\{d, q\}$.

Theorem 6 Let $d \geq 1$ and $k \geq 2$ be integers, and let X be a set of $k(d+1)$ points in $\mathbb{R}^d$ in general position with respect to the origin 0. Then the following properties hold for $B_0(X)$:

1. $B_0(X)$ is even;
2. $B_0(X) > 0 \Longrightarrow B_0(X) \geq k!$

Theorem 7 Let X be a set of $(d+1)(r-1)+1$ points in general position in $\mathbb{R}^d$, then the following properties hold for the number $T(X)$ of Tverberg partitions:

1. $T(X)$ is even for $q > d+1$
2. $T(X) \geq (q-d)!$

References

[1] Stephan Hell. "On the number of Birch partitions". In: *Discrete & Computational Geometry* 40.4 (2008), pp. 586–594.