Big line or big convex polygon handout

Jan Soukup

March 2025

1 Definitions and cups-caps theorems

Given an n-element point set P in the plane, we say that P is in *convex position* if P is the vertex set of a convex n-gon. We say that P is in *general position* if no three members of P are collinear.

Definition 1. *Let $ES_\ell(n)$ be the minimum N such that every N-point set in the plane contains either ℓ collinear points or n points in convex position.*

Let X be a k-element point set in the plane with distinct x-coordinates. We say that X forms a *k-cup* (*k-cap*) if X is in convex position and its convex hull is bounded above (below) by a single edge.

Definition 2. *Write $f_\ell(m,n)$ for the minimum N such that every N-point set in the plane contains either ℓ collinear members, an m-cup or an n-cap.*

Theorem 1. *There is an absolute constant $c > 1$ such that, for $m, n, \ell \geq 3$,*

$$f_\ell(m,n) \leq c(\min\{m-1, n-1\} + \ell) \cdot \binom{m+n-4}{n-2}.$$

Proof. Using the following lemma □

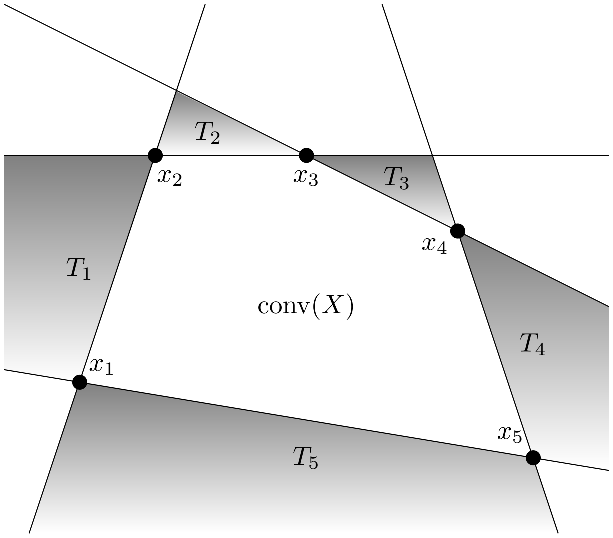
Lemma 2 (Beck 83'). *There is an absolute constant $\varepsilon > 0$ such that every t-element point set in the plane contains either εt collinear points or determines at least $\varepsilon\binom{t}{2}$ distinct lines.*

Theorem 3.

$$f_\ell(m,n) > \frac{\ell-1}{2}\binom{m+n-4}{n-2} - \frac{\ell-3}{2}\binom{m+n-6}{n-3}.$$

Proof. Proof by construction. □

Theorem 4. *There is a constant c_1 such that the following holds. Let P be an N-point planar set with no ℓ points on a line and $N > c_1\ell \cdot 2^{32k}$. Then there is a k-element subset $X \subset P$ that is either a k-cup or a k-cap such that, for the regions $T_1, \dots, T_{k-1}$ from the support (definition by picture) of X, the point sets $P_i = P \cap T_i$ satisfy $|P_i| \geq N/2^{32k}$. In particular, every $(k-1)$-tuple obtained by selecting one point from each P_i, $i = 1, \dots, k-1$, is in convex position.*



Proof. Proof by the following lemma. □

Lemma 5 (Chan 12', Matousek 92')**.** *Let P be a set of N points in the plane. Then, for any integer $r > 0$, there are disjoint subsets $P_1, \ldots, P_r$ of P and disjoint cells $\Delta_1, \ldots, \Delta_r$ in $\mathbb{R}^2$, with $P_i \subset \Delta_i$, such that $|P_i| \geq N/(8r)$ and every line in the plane crosses at most $O(\sqrt{r})$ cells Δ_i.*

2 Main results

Theorem 6. *For each $\ell, n \geq 3$, $ES_\ell(n) \geq (3\ell - 1) \cdot 2^{n-5} + 1$.*

Proof. Proof by construction. □

Theorem 7. *There exists $C > 0$ such that, for each $\ell, n \geq 3$, $ES_\ell(n) \leq \ell^2 \cdot 2^{n+C\sqrt{n \log n}}$.*

Proof. Proof by Theorem 4 and by modification of Theorem 1 explained below. □

We will need the following more general version of Theorem 1. Let K be a convex set in the plane. Then we say that the point set P *avoids* K if the line generated by any two points in P is disjoint from K. We say that K and P are *separated* if there is a line that separates K and conv(P). Suppose now that K is a convex set in the plane, P is a finite point set that avoids K and K and P are separated. Then, given a subset $X \subset P$, we say that X is an *inner-cap* with respect to K if, for each point $x \in X$, there is a line that separates x from $(X \setminus \{x\}) \cup K$. Similarly, we say that $X \subset P$ is an *outer-cup* with respect to K if, for each point $x \in X$, there is a line that separates $x \cup K$ from $X \setminus \{x\}$.

Lemma 8. *There is an absolute constant $c > 0$ such that the following holds. Let K be a convex set in the plane and let P be a finite point set in the plane that avoids K. If K and P are separated and*

$$|P| \geq c(\min\{m-1, n-1\} + \ell) \cdot \binom{m+n-4}{n-2},$$

then P contains either ℓ collinear points, an outer-cup with respect to K of size m or an inner-cap with respect to K of size n.