OPTIMIZING THE CGMS UPPER BOUND ON RAMSEY NUMBERS

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1 DEFINITIONS

Definition 1. *The* Ramsey number $R(k,l)$ *is the smallest positive integer* N *such that in any red-blue coloring of the edges of the complete graph on* N *vertices there exists either a complete subgraph on* k *vertices with all edges colored red (a red* K_k*) or a complete subgraph on* l *vertices with all edges colored blue (a blue* K_l*). We call* $R(k,k)$ *the* diagonal *Ramsey number.*

Definition 2. *Let* X, Y *be two non-empty disjoint subsets of vertices of a complete graph with edges colored red and blue. We say that* (X,Y) *is a* candidate. *We say that a candidate* (X,Y) *is* (k,l,t)*-good if* $X \cup Y$ *contains a red* K_k *or* X *contains a blue* K_t *or* Y *contains a blue* K_l*.*

Definition 3. *The density of* (X,Y) *is* $d(X,Y) = \frac{e_R(X,Y)}{|X||Y|}$*, where* $e_R(X,Y)$ *denotes the number of red edges with one endpoint in* X *and the other in* Y*. For some density* p*, we denote by* $f_p(X,Y) = e_R(X,Y) - p|X||Y|$ *the excess amount of red edges between* X *and* Y*.*

2 PREVIOUS RESULTS

Theorem 1. *[CGMS23] There exists a constant* $\varepsilon > 0$ *such that* $R(k,k) \le (4-\varepsilon)^k$*.*

Theorem 2. *[CGMS23]* $R(k,l) \le e^{-l/400+o(k)}\binom{k+l}{l}$*.*

3 RESULTS FROM PAPER

Theorem 3. *[GNNW24] For all positive integers* $k \ge l$*,*

$$R(k,l) \le 4(k+l)\left(\frac{(\sqrt{5}+1)(k+2l)}{4l}\right)^l \left(\frac{k+2l}{k}\right)^{k/2}.$$

Theorem 4. *[GNNW24] For all positive integers* $k \ge l$*,*

$$R(k,l) \le e^{G(l/k)k+o(k)}\binom{k+l}{l},$$

where $G(\lambda) = (-0.25\lambda + 0.03\lambda^2 + 0.08\lambda^3)e^{-\lambda}$*.*

Corollary 5. *It holds that*

$$R(k,k) \le (3.7992\ldots)^{k+o(k)}$$

and

$$R(k,l) \le e^{-l/20+o(k)}\binom{k+l}{l}.$$

References

[CGMS23] Marcelo Campos, Simon Griffiths, Robert Morris, and Julian Sahasrabudhe. An exponential improvement for diagonal Ramsey, 2023.

[GNNW24] Parth Gupta, Ndiame Ndiaye, Sergey Norin, and Louis Wei. Optimizing the CGMS upper bound on Ramsey numbers, 2024.