The asymptotics of r(4,t)

Sam Mattheus, Jacques Verstaete

Presented by Tomáš Hons

Definition 1. Let $r(s,t)$ be the smallest value such that any graph on at least $r(s,t)$ vertices contains either a clique of size s or an independent set of size t.

Theorem 1. *As $t \to \infty$,*

$$r(4,t) \in \Omega\left(\frac{t^3}{(\log t)^4}\right).$$

Proof overview.

- Using the magical properties of FPP, we obtain a graph H_q with
  - $n = q^2(q^2 - q + 1)$ vertices,
  - edges are a union of $q^3 + 1$ edge-disjoint cliques of order q^2,
  - each copy of K_4 in H_q has at least three vertices in one of these cliques.
- We consider the random n-vertex graph H_q^* as a union of complete bipartite subgraphs of the cliques of H_q (hence, H_q^* is K_4-free).
- There is an instance G_q^* of H_q^* with at least $2^{40}q^3$ edges; using the container method, it has at most $(q/\log^2 q)^t$ independent sets of size $t = 2^{30} q \log^2 q$.
- Thus, by sampling vertices with probability $(\log^2 q)/q$, we obtain a graph with at least $(q^3 \log^2 q)/2$ vertices and no independent sets of size t, yielding $r(4,t) \geq ct^3/\log^4 t$.

□

Theorem 2. *For q a power of a prime, there is a graph H_q with the following properties:*

(i) $|V(H_q)| = q^2(q^2 - q + 1)$,

(ii) there is a set $\mathcal{C}$ of $q^3 + 1$ maximal cliques of order q^2, every two sharing exactly one vertex,

(iii) each vertex lies in exactly $q + 1$ cliques of $\mathcal{C}$,

(iv) every copy of K_4 in H_q contains at least three vertices in some clique of $\mathcal{C}$.

Moreover, for each $X \subseteq V(H_q)$ of size $2^{24}q^2$, many edges of $H_q[X]$ lie in many cliques.

Theorem 3. *There is a realization of G_q^* of H_q^* such that for every set $X \subseteq V(G_q^*)$ of size at least $2^{24}q^2$,*

$$e(G_q^*[X]) \geq \frac{|X|^2}{256q}.$$

Proof of Theorem 1.

- By applying the container method appropriately, we obtain that G_q^* has at most $(q/\log^2 q)^t$ independent sets of size $t = 2^{30} q \log^2 q$.
- Randomly sample a set V of vertices of G with probability $\log^2 q/q$ independently for each vertex.
- Then the expected number of independent sets of size t is at most 1.
- It follows that there is a K_4-free graph with

  $$\frac{q^3 \log^2 q}{2} \geq c\frac{t^3}{\log^4 t}$$

  vertices and no independent set of size t.

□

Definition 2. We say that G is an (n, d, λ)-graph if

(i) $|V(G)| = n$,

(ii) G is d-regular,

(iii) the second largest eigenvalue (in the absolute value) is λ.

Theorem 4 (Expander mixing lemma)**.** *Let G be an (n, d, λ)-graph and let $X \subseteq V(G)$. Then*

$$\left|2e(X) - \frac{d}{n}|X|^2\right| \leq \lambda|X|.$$

If (n, d, λ)-graph satisfies $\lambda = O(\sqrt{d})$ (best possible), we call it *spectrally extremal graph.*

Theorem 5 (Sudakov, Szabo and Vu)**.** *Let G be a K_s-free (n, d, λ)-graph. Then*

$$d \in O\left(\lambda^{\frac{1}{s-1}} n^{1-\frac{1}{s-1}}\right).$$

If G is spectrally extremal, this gives

$$d \in O\left(n^{1-\frac{1}{2s-3}}\right).$$

Theorem 6 (Mubayi, Verstaete)**.** *If there exists a spectrally extremal K_s-free (n, d, λ)-graph with $d \in O(n^{1-1/(2s-3)})$, then*

$$r(s,t) \in \Omega\left(\frac{t^{s-1}}{(\log t)^{2s-4}}\right).$$

It is known

$$r(s,t) \leq (1 + o(1))\frac{t^{s-1}}{(\log t)^{s-2}}$$

by Ajtai, Komlós, Szemerédi.

Proposition 7. *If G is an (n, d, λ)-graph, then the number of independent sets of size $t \geq 2n(\log n)^2/d$ is at most*

$$\left(\frac{4e^2\lambda}{\log^2 n}\right)^t.$$

Conjecture 1. There is $C > 0$ such that the number of independent sets of size $t \geq C(2n \log n)/d$ in G is at most

$$\left(\frac{C\lambda}{\log n}\right)^t.$$

If Conjecture 1 true, then Theorem 6 improves to give (conditionally) the asymptotically matching lower bound

$$r(s,t) \in \Omega\left(\frac{t^{s-1}}{(\log t)^{s-2}}\right).$$