Forbidden acyclic patterns in 0-1 matrices

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The problem

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A does not avoid P = A contains P

- Extremal function:
  $\mathrm{Ex}(P, n) =$ maximum weight of an $n \times n$ 0-1 matrix avoiding P

# of 1 entries

connection to extremal graph theory

adjacency matrices	$\sim$	bipartite graphs
contains	$\sim$	contains as a subgraph
Ex(P, n)	=	ex(G, n) Turán-type extremal graph theory

connection to extremal graph theory

adjacency matrices	∼	bipartite graphs with vertex order
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Large number of applications in combinatorial geometry and other combinatorics, also in analysis of algorithms

Brief history

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$$\mathrm{Ex}(P_k, n) = \Omega(n^{4/3})$$

Bipartite adjacency matrix of C_{2k}

$$\mathrm{ex}(C_{2k}, n) = O\left(n^{1+\frac{1}{k}}\right)$$

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- Janzer, Janzer, Magnan, Methuku 2024: Every row of P has $\leq k$ 1's

  à la Füredi's result on bipartite graphs with each degree on one side $\leq k$

  $$\Longrightarrow \quad \text{Ex}(P_k, n) = O\left(n^{2-\frac{1}{k}+o(1)}\right)$$

acyclic patterns

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  Conj. 2' holds if P is built by simple rules

  e.g. adding first/last rows/column with a single 1

$\begin{pmatrix} & 1 & 1 \\ 1 & & 1 \\ & 1 & \end{pmatrix}$ ✅ $S = \begin{pmatrix} 1 & & 1 & \\ 1 & & & 1 \\ & 1 & & 1 \end{pmatrix}$ ❌ $S' = \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & & 1 \end{pmatrix}$ ❌

Smallest examples

acyclic pattern upper bounds

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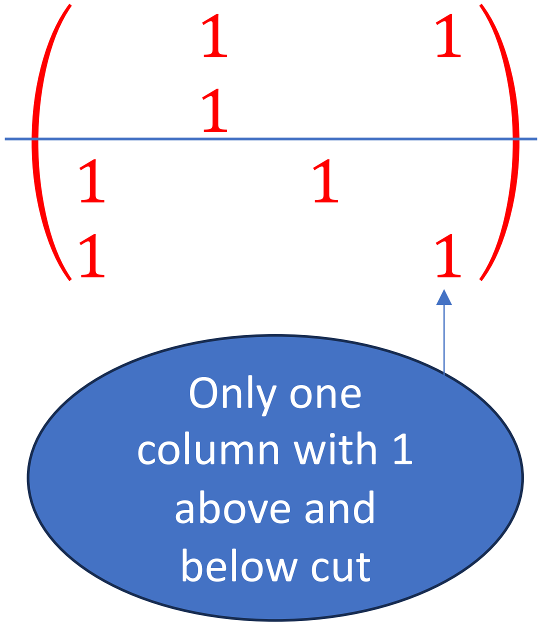
- Korándi, T., Tomon, Weidert 2019:

Conj. 2’’ $\mathrm{Ex}(P,n) = n^{1+o(1)}$ for acyclic P

Conj 2’’ holds if P is built from rows by stacking

Stacking:



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$$S = \begin{pmatrix} 1 & & 1 & \\ 1 & & & 1 \\ & 1 & & 1 \end{pmatrix} ✅ \qquad S' = \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & & 1 \end{pmatrix} ✅$$

$$\mathrm{Ex}(S,n) = n 2^{O(\sqrt{\log n})}$$

$$\mathrm{Ex}(S',n) = n 2^{O((\log n)^{2/3})}$$

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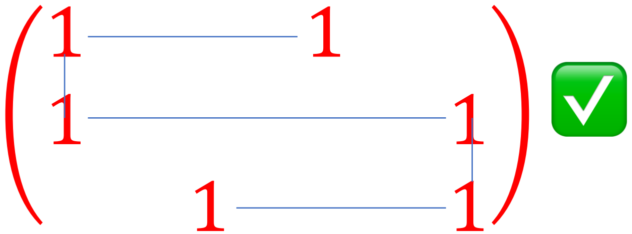
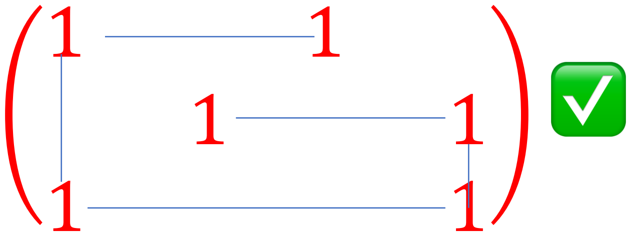
$S =$ 

$S' =$ 

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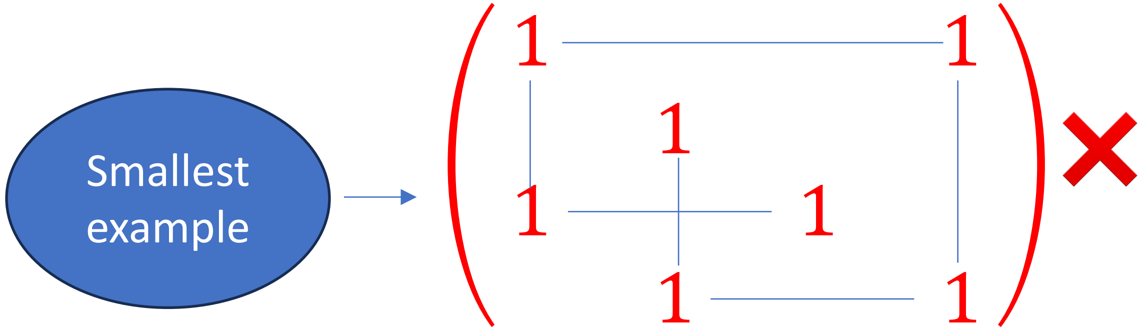

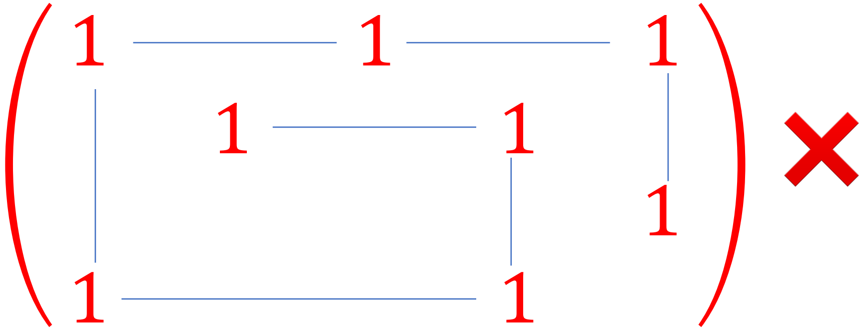

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Not known:

$\exists \varepsilon > 0 \;\; \forall$ acyclic P $\mathrm{Ex}(P,n) = O(n^{2-\varepsilon})$

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  $\text{Ex}(S',n) = n2^{\Omega(\sqrt{\log n})}$

Conj 2 really false

Conj 2' false!

upper bound for pattern S tight!

construction

$$P = \begin{pmatrix} & 1 & 1 \\ 1 & & 1 \end{pmatrix}$$

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construction

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Construction of a 0-1 matrix avoiding P:

rows and columns: $\{0,1\}^m$, ordered lexicographically

1 entry in row r, column c $\Leftrightarrow$ r and c differ in a single coordinate i

$$r_i = 0, \;\; c_i = 1$$

construction

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$n = 2^m$ rows / columns

$m2^{m-1} = \frac{1}{2} n \log n$ 1 entries

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To increase weight, we need to allow r and s differ in more positions

That may lead to large all-1 submatrix
⇒ contains every pattern

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To increase weight, we need to allow r and s differ in more positions

That may lead to large all-1 submatrix
$\Rightarrow$ contains every pattern

Pettie, T 2024:
r and s differ in
$O(1)$ positions
2025: unbounded
# of positions

construction

$$S = \begin{pmatrix} 1 & & 1 & \\ 1 & & & 1 \\ & 1 & & 1 \end{pmatrix}$$

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Construction of a 0-1 matrix avoiding S:

rows: $[m]\times[m]^b$

columns: $[m]^b\times\{0,1\}^b$

} ordered lexicographically

1 entry in row (s,r), column (c,i) $\iff$ $r - c = si$

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Column knows where the change is
Row knows how big the change is

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Column knows where the change is
Row knows how big the change is

$n = m^{b+1}$ rows / $(2m)^b$ columns - choose $m = 2^b, n = 2^{b^2+b}$

$\approx \frac{m}{b} n \approx n2^{\Theta(\sqrt{\log n})}$ 1 entries ✔

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1 entry in row (s, r), column (c, i) $\Leftrightarrow$ $r - c = si$

(c, i) (c', i')

| |

(s, r) - 1 —— 1

$c < c'$, first non-0 of $c' - c$ is s

SIMPLE

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ASSUME

S contained

c c' c'' c'''

| | | |

s — 1 —— 1

s' — 1 —— 1

s'' — 1 —— 1

rows: $[m] \times [m]^b$
columns: $[m]^b \times \{0,1\}^b$
} ordered lexicographically

1 entry in row (s, r), column (c, i) $\iff$ $r - c = si$

SIMPLE

(c, i) (c', i')

(s, r) - 1 1

(s', r')- 1

$c < c'$, first non-0 of $c' - c$ is s

$s < s'$

ASSUME
S contained

c c' c'' c'''

s — 1 1
s' — 1 1
s'' — 1 1

$s < s' < s''$

First non-0 of $x = c'' - c$ is s

First non-0 of $y = c''' - c$ is s'

First non-0 of $z = c''' - c'$ is s''

rows: $[m]\times[m]^b$

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(c,i) (c',i')

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$s<s'$

ASSUME

S contained

c c' c'' c'''

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s' — 1 —— 1

s'' — 1 —— 1

$s<s'<s''$

First non-0 of $x=c''-c$ is s

First non-0 of $y=c'''-c$ is s'

First non-0 of $z=c'''-c'$ is s''

$x,z\leq y$

$x+z\geq y$

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1 entry in row (s,r), column (c,i) $\Leftrightarrow$ $r-c=si$

SIMPLE

(c,i) (c',i')

(s,r) - 1 — 1

(s',r') - 1

$c<c'$, first non-0 of $c'-c$ is s

$s<s'$

ASSUME S contained

c c' c'' c'''

s — 1 — 1
s' — 1 — 1
s'' — 1 — 1

$s<s'<s''$

First non-0 of $x=c''-c$ is s

First non-0 of $y=c'''-c$ is s'

First non-0 of $z=c'''-c'$ is s''

$x,z\le y$
$x+z\ge y$

Open problems for acyclic patterns P

- Conjecture 2'': $\text{Ex}(n,P) = n^{1+o(1)}$ for all acyclic P
- Prove this for certain acyclic patterns, e.g.:

$$\begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & & & 1 \\ 1 & & & 1 & \end{pmatrix}$$

- Prove a weaker version for all acyclic P, e.g.: $\text{Ex}(n,P) = O(n^{1.5})$
- Characterize P with $\text{Ex}(n,P) = O(n)$
  For connected patterns, see Füredi, Kostochka, Mubayi, and Verstraete
  No conjecture in general, but nice conjecture for light patterns.
- What extremal functions show up?
  E.g., can it be strictly between $\Theta(n\alpha(n))$ and $\Theta(n \log n)$?
  Yes, for pairs of patterns.
- Lot more about non-acyclic patterns, like $\begin{pmatrix} & 1 & 1 \\ 1 & & 1 \\ 1 & 1 & \end{pmatrix}$