



Definition 0.1. For any $\varepsilon > 0$, an n -vertex non-empty graph G is an ε -expander if for all $U \subseteq V(G)$ such that $|U| \leq n/2$ we have $|N(U)| > \varepsilon|U|$.

Lemma 0.2 (Diameter bound). Let $\varepsilon > 0$ and let G be an n -vertex ε -expander graph. Then

$$\text{diam}(G) \leq 2 \left\lceil \log_{1+\varepsilon} \frac{n}{2} \right\rceil.$$

1 Pass to expanders

1.1 Sublinear expander

Informally, a *sublinear expander* is an ε -expander where $\varepsilon \rightarrow 0$ as $n \rightarrow \infty$. Thinking of $\varepsilon = \frac{1}{\log n}$ is a good concrete choice to keep in mind.

Lemma 1.1. Any n -vertex $(1/\log n)$ -expander G contains a path of length $\Omega(n/\log n)$ and a cycle of length at least $\Omega(n/\log^2 n)$.

Definition 1.2 (Robust sublinear expander). A graph G on $n \geq 1$ vertices is called a robust sublinear expander if for every $0 \leq \varepsilon \leq 1$, every non-empty $U \subseteq V(G)$ of size $|U| \leq n^{1-\varepsilon}$, and every $F \subseteq E(G)$ with $|F| \leq (\varepsilon/4) d(G) |U|$, we have $|N_{G-F}(U)| \geq (\varepsilon/4)|U|$.

Lemma 1.3 (Pass-to-expander lemma). Every non-empty graph G contains a subgraph H which is a robust sublinear expander and has average degree

$$d(H) \geq \frac{\log |V(H)|}{\log |V(G)|} d(G).$$

1.2 Finding Topological Cliques

Definition 1.4. A topological clique is obtained from a clique by replacing each edge with a path whose internal vertices are all distinct. The original vertices are the anchor points; their number is the order of the topological clique.

Theorem 1.5 (Bollobás–Thomason; Komlós–Szemerédi). Any graph with average degree d contains a topological clique of order $\Omega(\sqrt{d})$.

We only prove a weaker version:

Theorem 1.6. Any n -vertex graph with average degree d contains a topological clique of order $\Omega(\sqrt{d}/\log^2 n)$.

1.3 Rainbow Cycles

Definition 1.7. A coloring of a graph is *rainbow* if every edge receives a distinct color.

Definition 1.8. A proper edge-coloring of G is a map $\chi : E(G) \rightarrow C$ such that no two incident edges share a color.

Question 1.9. Given a graph H , its n -vertex rainbow Turán number $\text{ex}^*(n, H)$ is the minimum number of edges e needed to guarantee that any properly edge-colored graph G with at least e edges contains a rainbow copy of H .

Theorem 1.10 (Alon–Bucić–Sauerermann–Zakharov–Zamir (2025)). *There exists $C > 0$ such that every properly edge-colored graph on $n \geq 3$ vertices with average degree at least $C \log n \log \log n$ contains a rainbow cycle.*

2 Decomposition to expanders

Definition 2.1. An n -vertex graph G is an (ε, s) -expander if for every $U \subseteq V(G)$ and $F \subseteq E(G)$ with $1 \leq |U| \leq \frac{2}{3}n$ and $|F| \leq s|U|$, we have $|N_{G-F}(U)| \geq \varepsilon|U|$.

Lemma 2.2 (Almost-decomposition lemma). *Given an n -vertex graph G and integer $s \geq 0$, one can delete up to $4sn \log n$ edges from G so that the remaining edges can be partitioned into graphs $G_1, \dots, G_r$ satisfying:*

- $\sum_i |G_i| \leq 2n$;
- each G_i is a $(1/(32 \log^2 |G_i|), s)$ -expander.

2.1 Erdős–Gallai Cycle Decomposition

Conjecture 2.3 (Erdős–Gallai). *Any n -vertex graph can be decomposed into $O(n)$ cycles and edges.*

Theorem 2.4 (Lovász (1968)). *Every n -vertex graph can be decomposed into at most $n/2$ paths and cycles.*

Corollary 2.5. *Every n -vertex graph can be decomposed into paths so that each vertex is an endpoint of at most two paths.*

Theorem 2.6 (Bucić–Montgomery (2024)). *Any n -vertex graph can be decomposed into $O(n \log^* n)$ cycles and edges.*

$$2.6 \iff 2.7 \iff 2.8$$

Theorem 2.7. *Any n -vertex graph of average degree d can be decomposed into $O(n)$ cycles and $n \log^{O(1)} d$ edges.*

Theorem 2.8. *Any n -vertex graph can be decomposed into $O(n)$ cycles and $n \log^{O(1)} n$ edges.*