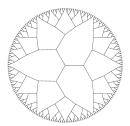
### Sublinear Expanders Lecture Notes by Matija Bucić



**Definition 0.1.** For any  $\varepsilon > 0$ , an n-vertex non-empty graph G is an  $\varepsilon$ -expander if for all  $U \subseteq V(G)$  such that  $|U| \le n/2$  we have  $|N(U)| > \varepsilon |U|$ .

**Lemma 0.2** (Diameter bound). Let  $\varepsilon > 0$  and let G be an n-vertex  $\varepsilon$ -expander graph. Then

$$\operatorname{diam}(G) \le 2 \left\lceil \log_{1+\varepsilon} \frac{n}{2} \right\rceil.$$

# 1 Pass to expanders

### 1.1 Sublinear expander

Informally, a *sublinear expander* is an  $\varepsilon$ -expander where  $\varepsilon \to 0$  as  $n \to \infty$ . Thinking of  $\varepsilon = \frac{1}{\log n}$  is a good concrete choice to keep in mind.

**Lemma 1.1.** Any n-vertex  $(1/\log n)$ -expander G contains a path of length  $\Omega(n/\log n)$  and a cycle of length at least  $\Omega(n/\log^2 n)$ .

**Definition 1.2** (Robust sublinear expander). A graph G on  $n \ge 1$  vertices is called a robust sublinear expander if for every  $0 \le \varepsilon \le 1$ , every non-empty  $U \subseteq V(G)$  of size  $|U| \le n^{1-\varepsilon}$ , and every  $F \subseteq E(G)$  with  $|F| \le (\varepsilon/4) d(G) |U|$ , we have  $|N_{G-F}(U)| \ge (\varepsilon/4) |U|$ .

**Lemma 1.3** (Pass-to-expander lemma). Every non-empty graph G contains a subgraph H which is a robust sublinear expander and has average degree

$$d(H) \ge \frac{\log |V(H)|}{\log |V(G)|} d(G).$$

## 1.2 Finding Topological Cliques

**Definition 1.4.** A topological clique is obtained from a clique by replacing each edge with a path whose internal vertices are all distinct. The original vertices are the anchor points; their number is the order of the topological clique.

**Theorem 1.5** (Bollobás–Thomason; Komlós–Szemerédi). Any graph with average degree d contains a topological clique of order  $\Omega(\sqrt{d})$ .

We only prove a weaker version:

**Theorem 1.6.** Any n-vertex graph with average degree d contains a topological clique of order  $\Omega(\sqrt{d}/\log^2 n)$ .

### 1.3 Rainbow Cycles

**Definition 1.7.** A coloring of a graph is rainbow if every edge receives a distinct color.

**Definition 1.8.** A proper edge-coloring of G is a map  $\chi: E(G) \to C$  such that no two incident edges share a color.

**Question 1.9.** Given a graph H, its n-vertex rainbow Turán number  $ex^*(n, H)$  is the minimum number of edges e needed to guarantee that any properly edge-colored graph G with at least e edges contains a rainbow copy of H.

**Theorem 1.10** (Alon–Bucić–Sauermann–Zakharov–Zamir (2025)). There exists C > 0 such that every properly edge-colored graph on  $n \geq 3$  vertices with average degree at least  $C \log n \log \log n$  contains a rainbow cycle.

# 2 Decomposition to expanders

**Definition 2.1.** An n-vertex graph G is an  $(\varepsilon, s)$ -expander if for every  $U \subseteq V(G)$  and  $F \subseteq E(G)$  with  $1 \leq |U| \leq \frac{2}{3}n$  and  $|F| \leq s|U|$ , we have  $|N_{G-F}(U)| \geq \varepsilon|U|$ .

**Lemma 2.2** (Almost-decomposition lemma). Given an n-vertex graph G and integer  $s \geq 0$ , one can delete up to  $4sn \log n$  edges from G so that the remaining edges can be partitioned into graphs  $G_1, \ldots, G_r$  satisfying:

- $\sum_{i} |G_i| \leq 2n$ ;
- each  $G_i$  is a  $(1/(32\log^2|G_i|), s)$ -expander.

### 2.1 Erdős–Gallai Cycle Decomposition

Conjecture 2.3 (Erdős-Gallai). Any n-vertex graph can be decomposed into O(n) cycles and edges.

**Theorem 2.4** (Lovász (1968)). Every n-vertex graph can be decomposed into at most n/2 paths and cycles.

Corollary 2.5. Every n-vertex graph can be decomposed into paths so that each vertex is an endpoint of at most two paths.

**Theorem 2.6** (Bucić–Montgomery (2024)). Any n-vertex graph can be decomposed into  $O(n \log^* n)$  cycles and edges.

$$2.6 \iff 2.7 \iff 2.8$$

**Theorem 2.7.** Any n-vertex graph of average degree d can be decomposed into O(n) cycles and  $n \log^{O(1)} d$  edges.

**Theorem 2.8.** Any n-vertex graph can be decomposed into O(n) cycles and  $n \log^{O(1)} n$  edges.