Timothy M. Chan, Isaac M. Hair: A Linear Time for the Maximum Overlap of Two Convex Polygons Under Translation (SoCG 2025)

presented by Giuseppe Pino

PhD Students Seminar, 13 November 2025

Problem 1. Given two convex polygons P and Q in the plane, with n and m edges respectively, find a vector $t \in \mathbb{R}^2$ that maximizes the area of $(P+t) \cap Q$, where $P+t$ denotes P translated by t.

1 Preliminaries

Lemma 1. $\sqrt{\text{Area}(P+t) \cap Q}$ *is downward concave over all values of* t *that yield nonzero overlap of* $P+t$ *with* Q.

Definition 1 (Configuration Space)**.** A polygonal chain $[v_i : v_j]$ for a polygon P is the union of consecutive edges between v_i and v_j.

Given two edges e, e' we define the parallelogram

$$\pi(e, e') = \{t \in \mathbb{R}^2 \mid e + t \cap e' \neq \emptyset\}.$$

Given two polygonal chains $A \subseteq P$ and $B \subseteq Q$ we define the set of all such line segments as

$$\Pi(A, B) = \bigcup_{e \in A, e' \in B} \partial\pi(e, e')$$

and define $\overset{\leftrightarrow}{\Pi}$ to be $\Pi(A, B)$ where each line segment is extended to a line.

Definition 2 (Blocking Scheme)**.** The angle of an edge e on the boundary of an arbitrary convex polygon A, denoted $\theta_A(e)$ is the angle (counterclockwise measured) between a horizontal rightward ray and e. For any subset X of edges of A define the angle range $\Lambda_A(X)$ to be the minimal interval in $[0, 2\pi)$ such that $\theta_A(e) \in \Lambda_A(X)$ for all $e \in X$. Given a parameter $b \in \mathbb{N}$, referred to as the block size, we define a partition of P and Q into blocks $P_1, \ldots, P_{\lceil N/b \rceil}$ and $Q_1, \ldots Q_{\lceil N/b \rceil}$. First, sort the set

$$\{\theta_P(e)\colon e \text{ is an edge of } P\} \cup \{\theta_Q(e)\colon e \text{ is an edge of } Q\}$$

in increasing order, and then partition the resulting sequence into consecutive subsequences $S_1 \ldots S_{\lfloor N/b \rfloor}$ of length b.

Definition 3 (The Block Structure)**.** Let $\mu \in \mathbb{R}^+$, $b \in \mathbb{N}$ and $\mathcal{T}$ be either a triangle, a line segment or a point. A $(\mu, b, \mathcal{T})$-block structure is a data structure containing the following:

1. Access to the vertices of P and Q in counterclockwise order with $\mathcal{O}(1)$ query time, and access to the area prefix sums for P and Q with $\mathcal{O}(1)$ query time.
2. A block size parameter b, which implies blocks $P_1, \ldots, P_{\lceil N/b \rceil}$ and $Q_1, \ldots Q_{\lceil N/b \rceil}$.
3. A partition of $S = \lceil N/b \rceil$ into subsets S_{good} and S_{bad} with the requirement that $|S_{bad}| \leq \mu$.
4. For every $i \in S_{good}$, access in time $\mathcal{O}(1)$ to a constant-complexity quadratic function f_i such that $f_i(t) = \text{Area}((P_i + t) \cap Q)$ for all $t \in \mathcal{T}$.

2 Subroutines

Problem 2 (MaxRegion)**.** Given a $(\mu, b, \mathcal{T})$-block structure for P and Q, find

$$\max_{t \in \mathcal{T}} \text{Area}((P+t) \cap Q)$$

along with the translation $t^* \in \mathcal{T}$ realizing the maximum.

Problem 3 (Cascade)**.** Given a $(\mu, b, \mathcal{T})$-block structure, parameters μ' and b' such that b divides b', and triangle or line segment $\mathcal{T}' \subseteq \mathcal{T}$, either produce a $(\mu', b', \mathcal{T}')$-block structure or report failure. Failure may be reported only if the interior of $\mathcal{T}'$ intersects more than $\mu'/2$ lines from the set

$$\mathcal{S} := \bigcup_{k \in \{0, \ldots, \lceil N/b' \rceil - 1\}} \left(\bigcup_{(i,j) \in \left[\frac{b'}{b}\right] \times \left[\frac{b'}{b}\right]} \overset{\leftrightarrow}{\Pi} \left(P_{k \cdot \frac{b'}{b} + i}, Q_{k \cdot \frac{b'}{b} + j} \right) \right)$$

where $P_1, \ldots, P_{\lceil N/b \rceil}$ and $Q_1, \ldots Q_{\lceil N/b \rceil}$ are the blocks of P and Q determined by b.

We denote the expected time complexity of the fastest algorithm for MaxRegion by $T_{d-\textsc{Max}}(N, b, \mu)$, where d is the dimension of the region $\mathcal{T}$. We write the expected time complexity of the fastest algorithm for Cascade as $T_{\textsc{Cascade}}(N, b, b', \mu)$.

Lemma 2 (Point Oracle)**.** *For all $2 \leq b \leq N$, for all μ,*

$$T_{0-\textsc{Max}}(N, b, \mu) = \mathcal{O}\left(N \cdot \frac{\log^2 b}{b} + \mu b \right).$$

Lemma 3 (Cascading Subroutine)**.** *For all $2 \leq b, b' \leq N$ such that b divides b', for all μ,*

$$T_{\textsc{Cascade}}(N, b, b', \mu) = \mathcal{O}\left(N \cdot \frac{\log b' \log b}{b} + \mu b^2 \right).$$

3 The Algorithm

Theorem 4. *For all $2 \leq b, b' \leq N^{o(1)}$ such that b divides b', for all μ, μ' such that $\mu' = N^{1-o(1)}$,*

$$\begin{aligned} T_{2-\textsc{Max}}(N, b, \mu) =& \mathcal{O}(\log(Nb'/\mu')) \cdot T_{1-\textsc{Max}}(N, b, \mu) + \mathcal{O}(N^{0.1+o(1)}) + \\ & \mathcal{O}(1) \cdot T_{\textsc{Cascade}}(N, b, b', \mu) + T_{2-\textsc{Max}}(N, b', \mu') \end{aligned}$$

$$\begin{aligned} T_{1-\textsc{Max}}(N, b, \mu) =& \mathcal{O}(\log(Nb'/\mu')) \cdot T_{0-\textsc{Max}}(N, b, \mu) + \mathcal{O}(N^{0.1+o(1)}) + \\ & \mathcal{O}(1) \cdot T_{\textsc{Cascade}}(N, b, b', \mu) + T_{1-\textsc{Max}}(N, b', \mu') \end{aligned}$$

Definition 5 (ε-cutting)**.** An ε-cutting for a set X of n hyperplanes in $\mathbb{R}^d$, for any $d \geq 1$, is a partition of $\mathbb{R}^d$ into simplices such that the interior of every simplex intersects at most $\varepsilon \cdot n$ hyperplanes of X.

Lemma 6. *Assume we can sample a uniformly random member of X in expected time $\mathcal{O}(n^{0.1})$. Then in $\mathcal{O}(n^{0.1+o(1)})$ expected time, we can find a set of $\mathcal{O}(1)$ simplices that constitutes an ε-cutting of X with high probability.*

Theorem 7. *There is an algorithm for Problem 1 running in expected time $\mathcal{O}(n + m)$.*

4 The Cascading Subroutine

Lemma 8. *Let X and Y be any two polygonal chains of P and Q, respectively. If $\Lambda_A(X) \cap \Lambda(Y) = \emptyset$, then X and Y intersect at most twice and we can find the intesection point(s) in time $\mathcal{O}(\log|X| \log|Y|)$.*

Lemma 9. *Let X and Y be polygonal chains of P and Q, respectively, and $\mathcal{T}'$ be a triangle or a line segment. If $\Lambda_P(X) \cap \Lambda_Q(Y) = \emptyset$, then we can determine if some line segment in $\Pi(X, Y)$ intersects the interior of $\mathcal{T}'$ in time $\mathcal{O}(\log|X| \log|Y|)$.*