Crossing Number of Simple 3-Plane Drawings

Goetze, Hoffmann, Rutter & Ueckerdt (GD 2025)
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Definitions and Notation

- A *drawing* Γ of a graph $G = (V, E)$ on the sphere is *simple* if the vertices are mapped to points, edges to Jordan arcs and the following holds:
  - any two edges meet only finitely many times; each meeting is either at a shared endpoint or a proper crossing;
  - no edge crosses itself, no two adjacent edges cross, and any two edges cross at most once.
- A drawing is *non-homotopic* if no "lens" formed by two parallel edges is empty.
- A drawing is *3-plane* if every edge is crossed at most 3 times.
- Let X be the set of crossings of Γ. For $i = 0, 1, 2, 3$, let $E_i \subseteq E$ be the set of edges with exactly i crossings; and $E_\times = E_1 \cup E_2 \cup E_3$.
- Replace each crossing by a degree-4 vertex; subdivide each edge into edge-segments. Removing all vertices and edges yields a decomposition of the sphere into cells. Each cell c has a boundary δc alternating between edge-segments and vertices or crossings.
  - The *size* of a cell, $\|c\|$, is the number of vertex-incidences plus number of edge-segment-incidences on δc (so we do not count crossings).
  - Denote by C the set of all cells, and by $C_a = \{\, c \in C : \|c\| = a\}$.
- A 3-plane drawing Γ is called *3-saturated* if it is connected, non-homotopic, and filled (for every cell c of Γ and every two distinct vertices u, v that lie on ∂c, the uncrossed edge uv appears on ∂c).

Main Result

Theorem 1 *Let Γ be any non-homotopic 3-plane drawing of a graph on $n \geq 3$ vertices. Then*

$$\textit{(i)}\ |E| \leq 5.5\,(n-2), \qquad \textit{(ii)}\ |X| \leq 5.5\,(n-2).$$

Remark 1 *The bound $5.5(n-2)$ is tight up to an additive constant.*

Proof Tools

Theorem 2 (Density Formula) *Let Γ be a connected drawing with at least one edge, and let C be the set of cells of Γ. Then for every real number t,*

$$|E| \;=\; t\,(|V|-2) \;-\; \sum_{c\in C} \frac{t-1}{4}\,(\,\|c\| - t\,) \;-\; |X|.$$

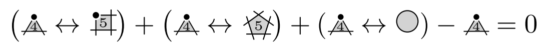
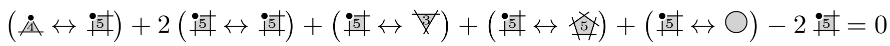
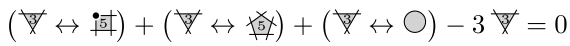
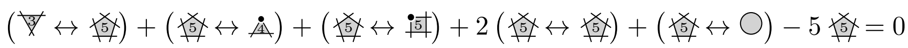
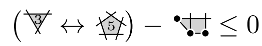
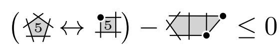
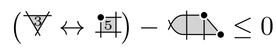
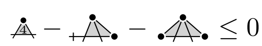
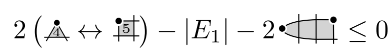
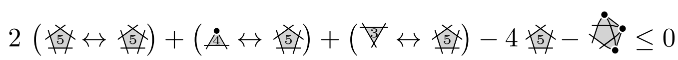
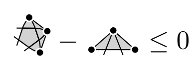
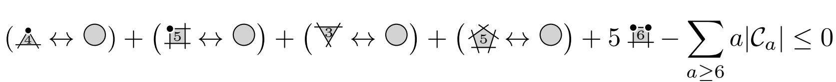
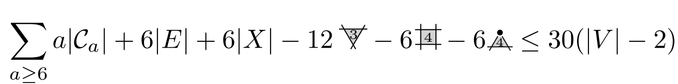
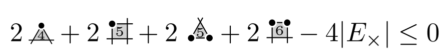
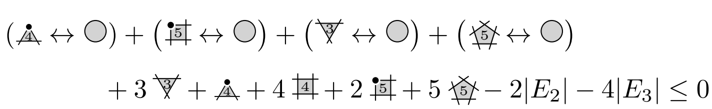
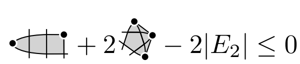
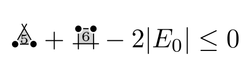
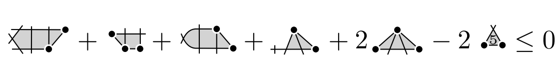
	(In)equality	$\lvert E\rvert$	$\lvert X\rvert$
(2A)		$\frac{-5}{16}$	$\frac{-7}{16}$
(2B)		$\frac{5}{16}$	$\frac{5}{16}$
(2C)		$\frac{-11}{24}$	$\frac{-11}{24}$
(2D)		$\frac{1}{8}$	$\frac{-3}{8}$
(3A)		$\frac{7}{48}$	$\frac{1}{48}$
(3B)		0	$\frac{1}{16}$
(3C)		$\frac{3}{16}$	$\frac{7}{48}$
(3D)		$\frac{3}{16}$	$\frac{5}{16}$
(3E)		0	$\frac{1}{16}$
(4A)		$\frac{3}{16}$	$\frac{13}{16}$
(4B)		$\frac{3}{16}$	$\frac{5}{16}$
(5A)		$\frac{11}{60}$	$\frac{11}{60}$
(5B)		$\frac{11}{60}$	$\frac{11}{60}$
(6)		$\frac{13}{80}$	$\frac{3}{80}$
(7)		$\frac{11}{40}$	$\frac{11}{40}$
(8A)	$\lvert E_1\rvert + \lvert E_2\rvert + \lvert E_3\rvert - \lvert E_\times\rvert = 0$	$\frac{-11}{20}$	$\frac{19}{20}$
(8B)	$\lvert E_1\rvert + 2\lvert E_2\rvert + 3\lvert E_3\rvert - 2\lvert X\rvert = 0$	$\frac{11}{20}$	$\frac{1}{20}$
(8C)		0	$\frac{1}{4}$
(9A)	$\lvert E_\times\rvert + \lvert E_0\rvert - \lvert E\rvert = 0$	$\frac{1}{10}$	$\frac{11}{10}$
(9B)		$\frac{1}{20}$	$\frac{11}{20}$
(9C)		$\frac{3}{16}$	$\frac{5}{16}$

Figure 1: Certificates for the upper bound on the number of edges and crossings in 3-saturated drawings in terms of the number of vertices. Each row corresponds to one inequality.