The sandglass conjecture beyond cancellative pairs

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Definition 1 (Recovering pair).
Given a finite set X and two families $\mathcal{A}, \mathcal{B} \subseteq 2^X$, we say that $(\mathcal{A}, \mathcal{B})$ is a ***recovering pair*** *over X if for all $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$,*

$$A_1 \backslash B_1 = A_2 \backslash B_2 \Longrightarrow A_1 = A_2$$
$$B_1 \backslash A_1 = B_2 \backslash A_2 \Longrightarrow B_1 = B_2.$$

Example 1.
Fix $S \subseteq X$, take $\mathcal{A} = \{A : S \subseteq A\}$, and $\mathcal{B} = \{B : B \subseteq S\}$. Then, the pair $(\{A^c : A \in \mathcal{A}\}, \mathcal{B})$ is recovering pair over X.

Conjecture 1 (The Sandglass Conjecture (posed by Ahlswede and Simonyi in 1994).
If $(\mathcal{A}, \mathcal{B})$ is a recovering pair over $[n]$, then $|\mathcal{A}||\mathcal{B}| \leq 2^n$.

Definition 2 (Optimal rate).

$$\mu_{rec} \coloneqq \lim_{n\to\infty} \max\{(|\mathcal{A}||\mathcal{B}|)^{1/n} : (\mathcal{A}, \mathcal{B}) \text{ is a recovering pair over } [n]\},$$
$$\mu_{can} \coloneqq \lim_{n\to\infty} \max\{(|\mathcal{A}||\mathcal{B}|)^{1/n} : (\mathcal{A}, \mathcal{B}) \text{ is a cancellative pair over } [n]\}.$$

Theorem 1 (Main Theorem).
Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over $[n]$, then

$$|\mathcal{A}||\mathcal{B}| \leq \max\{2.2499, \theta^\alpha \cdot \mu_{can}^{1-\alpha}\}^n$$

where $\alpha = 0.27$ and $\theta = 2.222$.

Corollary 1 (New upper bound (as a corollary of the Main Theorem)).
If $(\mathcal{A}, \mathcal{B})$ is a recovering pair over $[n]$, then $|\mathcal{A}||\mathcal{B}| \leq 2.2557^n$.

Definition 3 (Cancellative pair and k-uniform pair).
Given a finite set X and two families $\mathcal{A}, \mathcal{B} \subseteq 2^X$, we say that $(\mathcal{A}, \mathcal{B})$ is a ***cancellative pair*** *over X if for all $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$,*

$$A_1 \backslash B_1 = A_2 \backslash B_1 \Longrightarrow A_1 = A_2$$
$$B_1 \backslash A_1 = B_2 \backslash A_1 \Longrightarrow B_1 = B_2.$$

We say that a pair $(\mathcal{A}, \mathcal{B})$ over a set $[n]$ is ***k-uniform*** *if $|A| = |B| = k$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$.*

Lemma 1.
Suppose that for all $n, k \in \mathbb{N}$ with $k \leq n$, every k-uniform recovering pair over n-element ground set satisfies $|\mathcal{A}||\mathcal{B}| \leq \mu^n$ for some constant $\mu > 0$. Then the same inequality holds for any (not necessarily uniform) recovering pair.

Definition 4 (Filtered pair)**.**
*For the two familes $\mathcal{A}, \mathcal{B} \subseteq 2^X$, and for given $P \subseteq S \subseteq C \subset X$, the **filtered pair** $(\mathcal{A}_{C,S,P}, \mathcal{B}_{C,S})$ of the pair $(\mathcal{A}, \mathcal{B})$ is defined such that*

$$\mathcal{A}_{C,S,P} := \{A \backslash C : A \in \mathcal{A}, A \cap S = P\},$$
$$\mathcal{B}_{C,S} := \{B \backslash C : B \in \mathcal{B}, C \backslash B = S\}.$$

Proposition 1.
Let $(\mathcal{A}, \mathcal{B})$ be a cancellative pair over a set X. Then for any sets $P \subseteq S \subseteq C \subset X$ such that there is $B \in \mathcal{B}$ with $S = C \backslash B$, the filtered pair $(\mathcal{A}_{C,S,P}, \mathcal{B}_{C,S})$ satisfies

$$(|\mathcal{A}_{C,S,P}| =) \quad |\{A \backslash C : A \in \mathcal{A}, A \cap S = P\}| = |\{A \in \mathcal{A} : A \cap S = P\}|,$$
$$(|\mathcal{B}_{C,S}| =) \quad |\{B \backslash C : B \in \mathcal{B}, C \backslash B = S\}| = |\{B \in \mathcal{B} : C \backslash B = S\}|.$$

Proposition 2.
Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over a set X. Then for any sets $P \subseteq S \subseteq C \subseteq X$, the filtered pair $(\mathcal{A}_{C,S,P}, \mathcal{B}_{C,S})$ is cancellative over $X \backslash C$.

Define $\mathcal{F}_i := \{F \in \mathcal{F} : i \in F\}$, $\mathcal{F}'_i := \{F \in \mathcal{F} : i \notin F\}$ for a family $\mathcal{F}$ of subsets of X and $i \in X$ and we denote the size of each restriction by $a_i := |\mathcal{A}_i|/|\mathcal{A}|$, $a'_i := |\mathcal{A}'_i|/|\mathcal{A}|$, $b_i := |\mathcal{B}_i|/|\mathcal{B}|$, $b'_i := |\mathcal{B}'_i|/|\mathcal{B}|$.

Proposition 3 ($\mathcal{A}_{C,S,P}$ is reasonably sized)**.**
Let $(\mathcal{A}, \mathcal{B})$ be a recovering pair over X and let $S \subseteq C \subseteq X$ be such that $S = C \backslash B$ for some $B \in \mathcal{B}$. Then there exists a subset $P \subseteq S$ such that

$$\log_2(|\mathcal{A}_{C,S,P}|/|\mathcal{A}|) \geq -\sum_{i \in S} h(a_i),$$

where $h(p) := -p \log p - (1-p) \log(1-p)$.

Proposition 4.
Let $(\mathcal{A}, \mathcal{B})$ be a k-uniform recovering pair over X and suppose that there exists a constant θ such that for every $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $P \subseteq A \backslash B$ we have $|\mathcal{A}_{A,A\backslash B,P}||\mathcal{B}_{A,A\backslash B}| \leq \theta^{-k}|\mathcal{A}||\mathcal{B}|$. Then we have

$$\log_2 |\mathcal{A}| \leq \sum_{i \in X} f(a_i, b_i, \theta),$$

where $f(x, y, t) := x(1-y)h(x) + h(x(1-y)) - x \log_2 t$.

Corollary 2.
Let $(\mathcal{A}, \mathcal{B})$ be a k-uniform recovering pair over X and suppose that there exists a constant θ such that for every $A \in \mathcal{A}$, $B \in \mathcal{B}$ and for every $P_1 \subseteq A \backslash B$ and $P_2 \subseteq B \backslash A$ we have $|\mathcal{A}_{A,A\backslash B,P_1}||\mathcal{B}_{A,A\backslash B}| \leq \theta^{-k}|\mathcal{A}||\mathcal{B}|$, and $|\mathcal{B}_{B,B\backslash A,P_2}||\mathcal{A}_{B,B\backslash A}| \leq \theta^{-k}|\mathcal{A}||\mathcal{B}|$. Then

$$\log_2 |\mathcal{A}||\mathcal{B}| \leq \sum_{i \in X} g(a_i, b_i, \theta),$$

where $g(x, y, t) := f(x, y, t) + f(y, x, t)$.

Claim 1.
For $x, y \in (0, 1)$ and $\theta = 2.222$ we have $g(x, y, \theta) \leq \log_2(2.2499)$.