The Coarse Menger Theorems and Conjectures

Based on the work of T. Nguyen, A. Scott, and P. Seymour:

- *A counterexample to the coarse Menger conjecture*, JCTB (2025)
- *Asymptotic structure. IV. A counterexample to the weak coarse Menger conjecture* (2025)

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Menger's theorem (1927)

Let $k \geq 1$, G be a graph and let $S, T \subseteq V(G)$ be two sets of vertices. Then either

- there are k vertex-disjoint paths between S and T, or
- there is a set X of at most $k-1$ vertices such that every path between S and T contains a vertex of X.

Introduction

Strong Coarse Menger Conjecture

For all integers $k, d \geq 1$, there exists an integer $\ell > 0$, such that for any graph G and any $S, T \subseteq V(G)$, one of the following holds:

- There exist k paths between S and T that are pairwise at distance at least d.
- There exists a set $X \subseteq V(G)$ with $|X| \leq k - 1$ such that every path from S to T contains a vertex at distance at most ℓ from a member of X.

Weak Coarse Menger Conjecture

For all integers $k, d \geq 1$, there exist integers $m, \ell > 0$ such that for any graph G and any $S, T \subseteq V(G)$, one of the following holds:

- There exist k paths between S and T that are pairwise at distance at least d.
- There exists a set $X \subseteq V(G)$ with $|X| \leq m$ such that every path from S to T passes within distance at most ℓ from a member of X.

Conjecture	**Dist.(d)**	**Paths (k)**	**Separator ($\|X\|$)**	**Status**
Classic Menger				
Strong CMC				
Weak CMC				

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Conjecture	Dist.(d)	Paths (k)	Separator ($\|X\|$)	Status
Classic Menger	≥ 1	Any	$\leq k-1$	**TRUE**
Strong CMC				
Weak CMC				

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Conjecture	**Dist.(d)**	**Paths (k)**	**Separator ($\|X\|$)**	**Status**
Classic Menger	≥ 1	Any	$\leq k-1$	**TRUE**
Strong CMC				
	≥ 3	≥ 3	$\leq k-1$	**FALSE**
Weak CMC				

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Conjecture	Dist.(d)	Paths (k)	Separator ($\|X\|$)	Status
Classic Menger	≥ 1	Any	$\leq k-1$	**TRUE**
Strong CMC				
	≥ 3	≥ 3	$\leq k-1$	**FALSE**
Weak CMC	≥ 3	≥ 3	any const. m	**FALSE**

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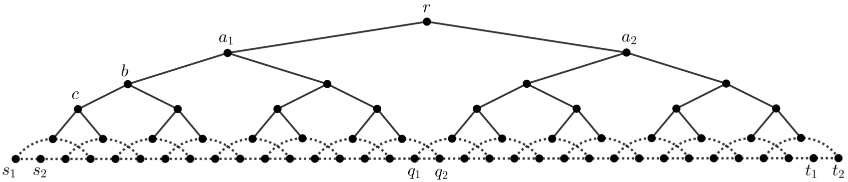
Conjecture	Dist.(d)	Paths (k)	Separator ($\|X\|$)	Status
Classic Menger	≥ 1	Any	$\leq k-1$	**TRUE**
Strong CMC	≥ 2	2	≤ 1	**TRUE**
	≥ 3	≥ 3	$\leq k-1$	**FALSE**
Weak CMC	≥ 3	≥ 3	any const. m	**FALSE**

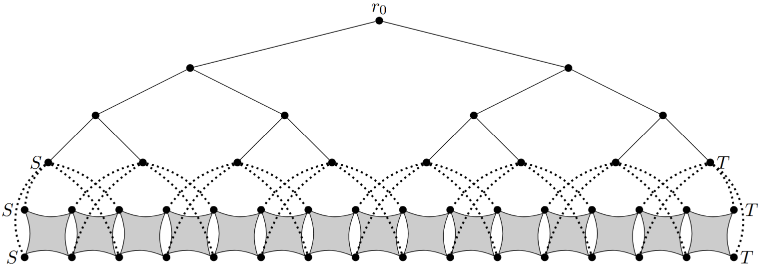
Conjecture	**Dist.(d)**	**Paths (k)**	**Separator ($\|X\|$)**	**Status**
Classic Menger	≥ 1	Any	$\leq k-1$	**TRUE**
Strong CMC	≥ 2	2	≤ 1	**TRUE**
	2	≥ 3	$\leq k-1$	**OPEN**
	≥ 3	≥ 3	$\leq k-1$	**FALSE**
Weak CMC	≥ 3	≥ 3	any const. m	**FALSE**

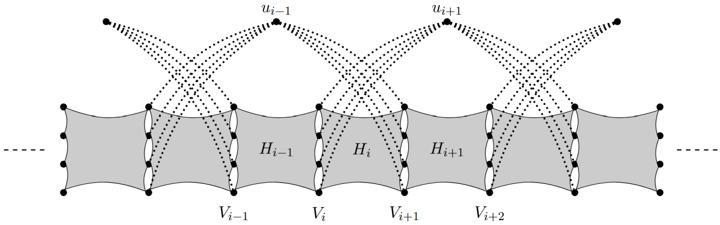
For The Strong Coarse Menger Conjecture ($k = d = 3$):



The Counterexamples

For The Weak Coarse Menger Conjecture ($k = d = 3$):





The Powerful Intervals

- **Def:** An **interval** is a pair (a, b) of integers with $a \leq b$, and its length is $b - a$.
- **Def:** If H is a set of intervals that can be written

$$H = \{(a_i, b_i) : 1 \leq i \leq t)\}$$

  for some $t \geq 1$, such that $0 \leq a_1 \leq a_2 \leq \cdots \leq a_t \leq n$ and $0 \leq b_1 \leq b_2 \leq \cdots \leq b_t \leq n$, we call this the **standard form of** H ($\iff$ there do not exist distinct $(a, b), (c, d) \in H$, such that $a \leq c \leq d \leq b$).
- **Def:** If $(a, b), (c, d)$ are intervals, we say (a, b) **captures** (c, d) if $a \leq c \leq d \leq b$; and a set H of intervals captures (c, d) if there exists $(a, b) \in H$ that captures (c, d).
- **Def:** A set of intervals H is ℓ-**powerful** in $(0, n)$ if it captures every interval of length ℓ that is contained within $(0, n)$.

The Powerful Intervals

Lemma 2.4

If $0 < \ell \leq n$ and H is a set of intervals that is ℓ-powerful in $(0, n)$, and minimal with this property, then in standard form $a_j \geq b_i - \ell + 2$ for all $i, j \in \{1, \ldots, t\}$ with $j \geq i + 2$.

Lemma 2.5

If $0 < 2\ell \leq n$ and H is a set of intervals that is 2ℓ-powerful in $(0, n)$, then there exists $H' \subseteq H$, minimally ℓ-powerful in $(0, n)$, that can be written $H' = \{(a_i, b_i) : 1 \leq i \leq t\}$ in standard form, such that

- $a_j \geq b_i - \ell + 2$ for $1 \leq i, j \leq t$ with $j \geq i + 2$; and
- $b_i - b_{i-1} \geq \ell$ for $1 < i < t$, and $b_{i-1} - a_i \geq \ell$ for $1 < i \leq t$.

The Powerful Intervals

Lemma 2.6

If $0 < 4\ell \leq n$ and H is a set of intervals that is 4ℓ-powerful in $(0, n)$, then there exists $H' \subseteq H$, ℓ-powerful in $(0, n)$, which can be written $H' = \{(a_i, b_i) : 1 \leq i \leq t\}$ in standard form, such that the order of the numbers $a_1, \ldots, a_t$ and $b_1, \ldots, b_t$ is:

$$0 = a_1 < a_2 < \min(a_3, b_1) < \max(a_3, b_1) < \min(a_4, b_2) < \max(a_4, b_2) < \ldots$$
$$\cdots < \min(a_t, b_{t-2}) < \max(a_t, b_{t-2}) < b_{t-1} < b_t = n.$$

Every two of $a_1, \ldots, a_t, b_1, \ldots, b_t$ differ by at least ℓ, except possibly the pairs (a_1, a_2), (b_{t-1}, b_t), and (a_i, b_{i-2}) for $3 \leq i \leq t$.

The Proof of The Strong Coarse Menger Theorem ($k = d = 3$)

Theorem 2.7

Let G be a non-null graph and let $S, T \subseteq V(G)$. Then either:

- There are two paths between S, T with distance at least three; or
- There exists $x \in V(G)$ such that every $S - T$ path is at distance ≤ 161 from x.

The Proof of The Strong Coarse Menger Theorem $(k = 2, d = 3)$

- **Claim 1:** The set of intervals $\{(a(C), b(C)) : C \in \mathcal{C}\}$ is 16ℓ-powerful in $(0, n)$.
- **Claim 2:** For $1 \le i, j \le t$, if $j \ge i + 3$ then $d(D_i, D_j) \ge 4\ell - 2c + 2$.
- **Claim 3:** If $X \subseteq V(G)$ is connected and $|\Delta(X)| \ge 4$, then $|X| \ge 30$.
- **Claim 4:** For every joint X, $d(X, R) > c - (|X| - 1)/2$.
- **Claim 5:** Let F_1, F_2 be distinct supercomponents, and let Q be a path of G with ends $f_1 \in V(F_1)$ and $f_2 \in V(F_2)$.
  - If both f_1, f_2 belong to joints then $|Q| \ge 16$;
  - if exactly one of f_1, f_2 belongs to a joint X then $|Q| \ge 8$, and $|Q| \ge 24$ if $|\Delta(X)| \ge 3$; and
  - if neither of f_1, f_2 belong to joints then $|Q| \ge 6$.

  In any case, $d(F_1, F_2) \ge 5$.
- **Claim 6:** If F_1, F_2 are distinct supercomponents, and A is a path of length at most $c + 1$ between F_2 and R, then $d(F_1, A) \ge 3$.
- **Claim 7:** The order of the numbers $a_1, \dots, a_s$ and $b_1, \dots, b_s$ is:

  $$0 = a_1 < a_2 < \min(a_3, b_1) < \max(a_3, b_1) < \min(a_4, b_2) < \max(a_4, b_2) < \cdots$$
  $$\cdots < \min(a_s, b_{s-2}) < \max(a_s, b_{s-2}) < b_{s-1} < b_s = n.$$

  Every two of $a_1, \dots, a_s, b_1, \dots, b_s$ differ by at least ℓ, except possibly the pairs (a_1, a_2), (b_{s-1}, b_s), and (a_i, b_{i-2}) for $3 \le i \le s$.

Thank you for your attention!