The Coarse Menger Theorems and Conjectures

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Based on the work of T. Nguyen, A. Scott, and P. Seymour:

- *A counterexample to the coarse Menger conjecture*, JCTB (2025)
- *Asymptotic structure. IV. A counterexample to the weak coarse Menger conjecture* (2025)

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1 Menger's Theorem

Theorem 1.1 (**Menger's Theorem, 1927**). *Let $k \geq 1$, G be a graph and let $S, T \subseteq V(G)$ be two sets of vertices. Then either*

- *there are k vertex-disjoint paths between S and T, or*
- *there is a set X of at most $k-1$ vertices such that every path between S and T contains a vertex of X.*

2 The Coarse Menger Theorems and Conjectures

Conjecture 2.1 (**Strong Coarse Menger Conjecture**). *For all integers $k, d \geq 1$, there exists an integer $\ell > 0$, such that for any graph G and any $S, T \subseteq V(G)$, one of the following holds:*

- *There exist k paths between S and T that are pairwise at distance at least d.*
- *There exists a set $X \subseteq V(G)$ with $|X| \leq k-1$ such that every path from S to T contains a vertex at distance at most ℓ from a member of X.*

Conjecture 2.2 (**Weak Coarse Menger Conjecture**). *For all integers $k, d \geq 1$, there exist integers $m, \ell > 0$ such that for any graph G and any $S, T \subseteq V(G)$, one of the following holds:*

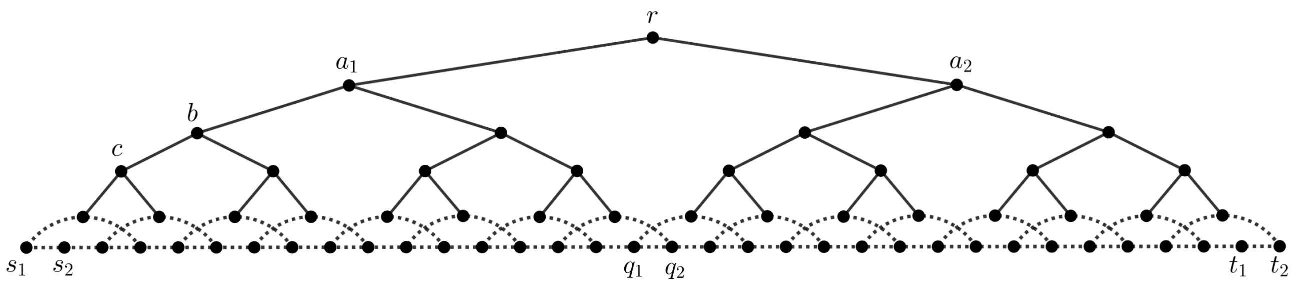
- *There exist k paths between S and T that are pairwise at distance at least d.*
- *There exists a set $X \subseteq V(G)$ with $|X| \leq m$ such that every path from S to T passes within distance at most ℓ from a member of X.*

2.1 The Current State of the Conjectures

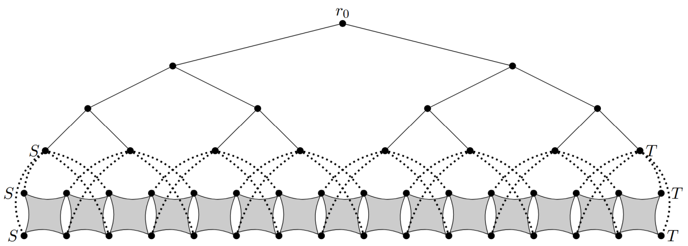
Conjecture	Dist.(d)	Paths (k)	Separator ($\|X\|$)	**Status**
Classic Menger	≥ 1	Any	$\leq k-1$	**TRUE**
Strong CMC	≥ 2	2	≤ 1	**TRUE**
	2	≥ 3	$\leq k-1$	**OPEN**
	≥ 3	≥ 3	$\leq k-1$	**FALSE**
Weak CMC	≥ 3	≥ 3	any const. m	**FALSE**

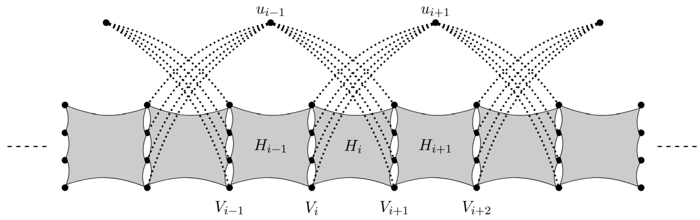
2.2 The Counterexamples

For The Strong Coarse Menger Conjecture ($k = d = 3$)**:**



For The Weak Coarse Menger Conjecture:





2.3 The Powerful Intervals

Definition 2.3. *A set of intervals H is **ℓ-powerful** in $(0, n)$ if it captures every interval of length ℓ that is contained within $(0, n)$.*

Lemma 2.4. *If $0 < \ell \leq n$ and H is a set of intervals that is ℓ-powerful in $(0, n)$, and minimal with this property, then in standard form $a_j \geq b_i - \ell + 2$ for all $i, j \in \{1, \ldots, t\}$ with $j \geq i + 2$.*

Lemma 2.5. *If $0 < 2\ell \leq n$ and H is a set of intervals that is 2ℓ-powerful in $(0, n)$, then there exists $H' \subseteq H$, minimally ℓ-powerful in $(0, n)$, that can be written $H' = \{(a_i, b_i) : 1 \leq i \leq t\}$ in standard form, such that*

- *$a_j \geq b_i - \ell + 2$ for $1 \leq i, j \leq t$ with $j \geq i + 2$; and*
- *$b_i - b_{i-1} \geq \ell$ for $1 < i < t$, and $b_{i-1} - a_i \geq \ell$ for $1 < i \leq t$.*

Lemma 2.6. *If $0 < 4\ell \leq n$ and H is a set of intervals that is 4ℓ-powerful in $(0, n)$, then there exists $H' \subseteq H$, ℓ-powerful in $(0, n)$, which can be written $H' = \{(a_i, b_i) : 1 \leq i \leq t\}$ in standard form, such that the order of the numbers $a_1, \ldots, a_t$ and $b_1, \ldots, b_t$ is:*

$$0 = a_1 < a_2 < \min(a_3, b_1) < \max(a_3, b_1) < \min(a_4, b_2) < \max(a_4, b_2) < \ldots$$
$$\cdots < \min(a_t, b_{t-2}) < \max(a_t, b_{t-2}) < b_{t-1} < b_t = n.$$

Every two of $a_1, \ldots, a_t, b_1, \ldots, b_t$ differ by at least ℓ, except possibly the pairs (a_1, a_2), (b_{t-1}, b_t), and (a_i, b_{i-2}) for $3 \leq i \leq t$.

2.4 The Proof of The Strong Coarse Menger Theorem ($k = 2, d = 3$)

Theorem 2.7. *Let G be a non-null graph and let $S, T \subseteq V(G)$. Then either:*

- *There are two paths between S, T with distance at least three; or*
- *There exists $x \in V(G)$ such that every $S - T$ path is at distance ≤ 161 from x.*

Notation 1. *Let G be a connected graph and $S, T \subseteq V(G)$ be two disjoint vertex sets. Denote*

- *R an $S-T$ path of minimum length, with vertices $r_1, \ldots, r_{n-1}$ where $r_1 \in S$ and $r_{n-1} \in T$*
- *$W = \{v \in V(G) \mid d(v, R) \leq c\}$ for an integer constant c The set of vertices with $d(v, R) = c$ is called the **surface***
- *$\mathcal{C}$ the set of all connected components of induced subgraph $G \setminus W$*
- *$\mathcal{D} = \{D_1, \ldots, D_t\}$ a specific subset of $\mathcal{C}$ chosen via Lemma 2.6*
- *Δ the union of the vertex sets of the members of $\mathcal{D}$*
- *$\Delta(X) = \{V(D) \mid D \in \mathcal{D}, X \cap N(D) \neq \emptyset\}$*

Claim 1. *The set of intervals $\{(a(C), b(C)) : C \in \mathcal{C}\}$ is 16ℓ-powerful in $(0, n)$.*

Claim 2. *For $1 \leq i, j \leq t$, if $j \geq i + 3$ then $d(D_i, D_j) \geq 4\ell - 2c + 2$.*

Claim 3. *If $X \subseteq V(G)$ is connected and $|\Delta(X)| \geq 4$, then $|X| \geq 30$.*

Definition 2.8. *Let us say a **joint** is a subset $X \subseteq V(G)$, inducing a connected subgraph, such that every vertex of $X \cap W$ has distance at most $(|X| - 1)/2$ from the surface, and either*

- *$|\Delta(X)| \geq 2$ and $|X| \leq 3$; or*
- *$|\Delta(X)| \geq 3$ and $|X| \leq 8$.*

Let Z be the union of all the joints.

Definition 2.9. *Each component of the subgraph $G[Z \cup \Delta]$ is called a **supercomponent**. Let $\mathcal{F}$ be the set of all supercomponents.*

Claim 4. *For every joint X, $d(X, R) > c - (|X| - 1)/2$.*

Claim 5. *Let F_1, F_2 be distinct supercomponents, and let Q be a path of G with ends $f_1 \in V(F_1)$ and $f_2 \in V(F_2)$.*

- *If both f_1, f_2 belong to joints then $|Q| \geq 16$;*
- *if exactly one of f_1, f_2 belongs to a joint X then $|Q| \geq 8$, and $|Q| \geq 24$ if $|\Delta(X)| \geq 3$; and*
- *if neither of f_1, f_2 belong to joints then $|Q| \geq 6$.*

In any case, $d(F_1, F_2) \geq 5$.

Claim 6. *If F_1, F_2 are distinct supercomponents, and A is a path of length at most $c+1$ between F_2 and R, then $d(F_1, A) \geq 3$.*

Claim 7. *The order of the numbers $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ is:*

$$\begin{aligned} &0 = a_1 < a_2 < \min(a_3, b_1) < \max(a_3, b_1) < \min(a_4, b_2) < \max(a_4, b_2) < \cdots \\ &\cdots < \min(a_s, b_{s-2}) < \max(a_s, b_{s-2}) < b_{s-1} < b_s = n. \end{aligned}$$

Every two of $a_1, \ldots, a_s, b_1, \ldots, b_s$ differ by at least ℓ, except possibly the pairs (a_1, a_2), (b_{s-1}, b_s), and (a_i, b_{i-2}) for $3 \leq i \leq s$.