Characterizing and Recognizing Twistedness (GD 2025 Best paper)

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Doctoral Seminar 18/12/2025

Basic definitions

Definition. *A* simple drawing *of a graph G is a drawing of G in the plane with vertices represented by distinct points and edges represented by Jordan arcs between end vertices through other vertices, and each pair of edges share at most one point (a proper crossing or a common endpoint) and no three edges cross.*

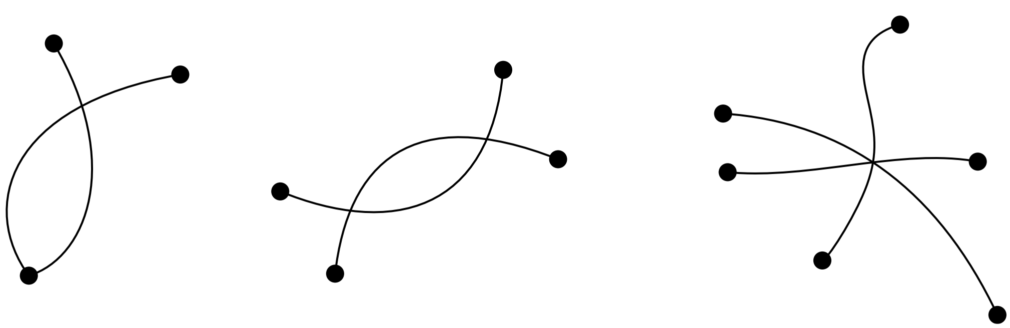

Figure 1: Forbidden patterns in simple drawings

Definition. *Two simple drawings are* strongly isomorphic *if there is some homeomorphism of the plane transforming one drawing into another.*

Definition. *A drawing D is a* generalized twisted drawing *if it is strongly isomorphic to a simple drawing in which there is a point O such that each ray emanating from O crosses each edge of the drawing at most once and there exists a ray r emanating from O such that all edges of the drawing cross r.*

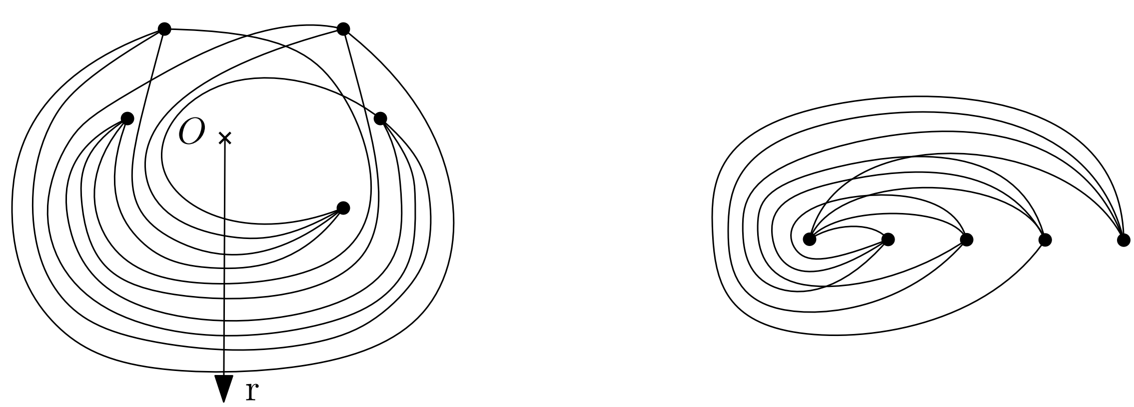

Figure 2: Two realizations of a twisted drawing of K_5

Definition. *Let D be a simple drawing and $v \in D$ be a vertex. The cyclic ordering of the edges incident to v in D is called* rotation at v. *The set of rotations of all vertices in D is called the* rotation system of D.

Definition. *For a graph G, an* abstract rotation system *of G is an assignment, to every $v \in G$, of a cyclic order of its adjacent vertices. An abstract rotation system is called* realizable *if there exists a simple drawing with that rotation system. It is called* generalized twisted *if the drawing is a generalized twisted drawing.*

1 Main results

Theorem 1. *Let R be an abstract rotation system of K_n with $n \geq 7$. Then R is generalized twisted if and only if every subrotation system induced by five vertices is generalized twisted.*

Theorem 2. *Let R be an abstract rotation system of K_n with $n \geq 7$. Then R is generalized twisted if and only if there are two vertices v_1, v_2 and a bipartition $A \cup B$ of $V \setminus \{v_1, v_2\}$ where some of A, B might be empty such that:*

1. *For every pair of vertices a and a' in A, the edge aa' crosses v_1v_2.*
2. *For every pair of vertices b and b' in B, the edge bb' crosses v_1v_2.*
3. *For every vertex $a \in A$ and every vertex $b \in B$, the edge ab does not cross v_1v_2.*
4. *Beginning at v_2, in the rotation of v_1, all vertices in B appear before all vertices in A.*
5. *Beginning at v_1, in the rotation of v_2, all vertices in B appear before all vertices in A.*

Theorem 3. *Let R be a realizable rotation system of K_n. It can be decided in time $O(n^2)$ if R is generalized twisted.*

2 Not so basic definitions and facts

From now on, D is a simple drawing of a complete graph K_n.

Definition. *Let $v \in D$ be a vertex. Then the* star *at v is the set of all edges adjacent to v and is denoted by $S(v)$.*

Definition. *A* triangle *Δ in D is a simple closed curve formed by tree vertices and the edges connecting them. A triangle Δ has divides the plane into two* sides.

Antipodal vi-cells A cell that is incident to a vertex (that is, has the vertex on its boundary) is called a *vi-cell.* Two cells are *antipodal* if they are on different sides in every triangle in D. Two antipodal vi-cells are called *via*-cells. A vertex adjacent to a via-cell is called a *via-vertex.*

Theorem 4 (Aichholzer et al., SoCG 2022)**.** *The rotation system of D is generalized twisted if and only if D contains a pair of antipodal vi-cells.*

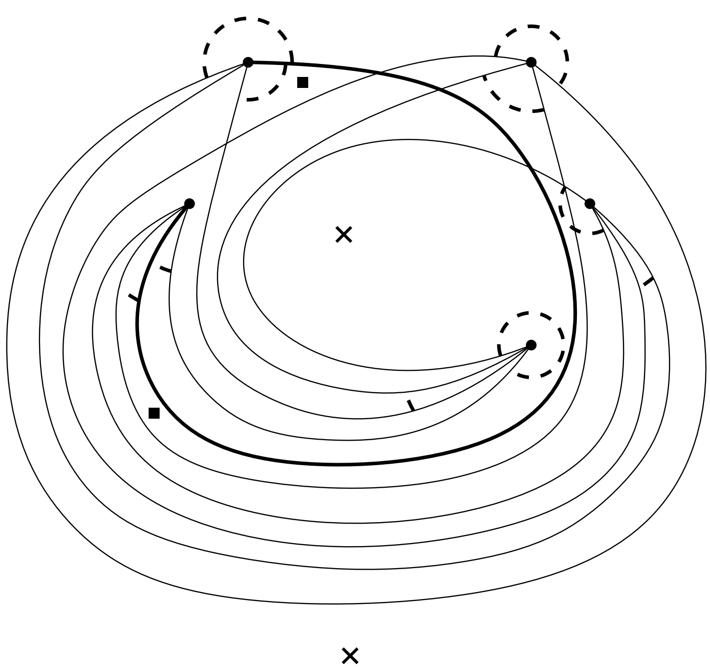

Figure 3: A generalized twisted drawing of K_5. Maximal crossing edge is marked bold, two pairs of via-cells are marked by squares and crosses, dashed circular arcs mark the empty triangles.

Empty star triangles A triangle Δ in D with vertices x, y, z is a *star-triangle at* x if $S(x)$ does not cross the edge yz. It is a *empty star triangle* if one of the sides of Δ contains no vertices of D. We call two star triangles *adjacent* if they share an edge and span different angles at x.

Lemma 5 (Aichholzer et al., Graphs and Combinatorics 2015)**.** *Let v be a vertex of a simple drawing D of K_n. Then D contains at least two empty star triangles at v. If vxy is a star triangle at v then it is empty if and only if the vertices x and y are consecutive in the rotation of v.*

Lemma 6 (García et al., GD 2022)**.** *Let D be a generalized twisted drawing of K_n and let v be a vertex of D. Then there are exactly two empty star triangles at v. If $(C_1, C_2$, is a pair of antipodal vi-cells in D, then one of the two empty star triangles at v contains C_1 on the empty side and the other triangle contains C_2 on the empty side.*

Maximum crossing edges An edge $e = xy$ is *maximum crossing* if it crosses every edge not adjacent to x and y.

Lemma 7. *Any simple drawing of K_n with $n \geq 5$ has at most one maximum crossing edge.*

Lemma 8. *Let D be a simple drawing of K_n that has a maximum crossing edge, v_1v_2. Let C_1 and C_1' be the two cells incident to v_1 along the edge v_1v_2 and let C_2 and C_2' be the two cells incident to v_2 along the edge v_1v_2 such that when going along the edge from v_1 to v_2, the cell C_1 is on the other side of v_1v_2 than the cell C_2 (and consequently C_1' is on the other side of v_1v_2 than C_2'). Then D is generalized twisted and (C_1, C_2) and (C_1', C_2') are two pairs of antipodal vi-cells.*