Selected Results from The Method of Hypergraph Containers

Sudatta Bhattacharya

Definitions and Theorems

Definition 1 (Extremal Number). *The extremal number of a graph H with respect to the Erdős-Rényi random graph $G(n,p)$ is defined to be*

$$ex(G(n,p),H) := \max\{e(G) : G \subseteq G(n,p) \text{ and } H \not\subseteq G\}.$$

Theorem 1 (Frankl and Rödl). *If $p \geq 1/\sqrt{n}$, then*

$$ex(G(n,p),K_3) = \left(\frac{1}{4} + o(1)\right) pn^2 \quad \text{with high probability as } n \to \infty.$$

Theorem 2 (The Container Theorem for Triangle-Free Graphs). *For each $n \in \mathbb{N}$, there exists a collection $\mathcal{G}$ of graphs on n vertices with the following properties:*

(a) $|\mathcal{G}| \leq n^{O(n^{3/2})}$.

(b) Each $G \in \mathcal{G}$ contains $o(n^3)$ triangles.

(c) Each triangle-free graph on n vertices is contained in some $G \in \mathcal{G}$.

Theorem 3 (Supersaturation for Triangles). *For every $\epsilon > 0$, there exists $\delta > 0$ such that the following holds. If G is a graph on n vertices with*

$$e(G) > \left(\frac{1}{4} + \epsilon\right) n^2,$$

then G has at least δn^3 triangles.

Definition 2 (Triangle Encoding Hypergraph). *The hypergraph encoding triangles in K_n is the 3-uniform hypergraph H with vertex set $V(H) = E(K_n)$ and edge set*

$$E(H) = \{\{f_1, f_2, f_3\} \subseteq E(K_n) : \{f_1, f_2, f_3\} = E(K_3)\}.$$

In words, the edges of H encode the triangles in K_n.

The Hypergraph Container Lemma for 3-uniform hypergraphs. For every $c > 0$, there exists $\delta > 0$ such that the following holds. Let H be a 3-uniform hypergraph with average degree d, set $\tau := 1/\sqrt{d}$, and suppose that $\tau \leq \delta$, and that

$$\Delta_1(H) \leq c \cdot d,$$
$$\Delta_2(H) \leq c \cdot \sqrt{d}.$$

Then there exists a collection $\mathcal{C}$ of subsets of $V(H)$, with

$$|\mathcal{C}| \leq \binom{v(H)}{\tau \cdot v(H)},$$

such that

(a) For every independent set $I \in I(H)$, there exists $C \in \mathcal{C}$ such that $I \subseteq C$.

(b) $|C| \leq (1-\delta)v(H)$ for every $C \in \mathcal{C}$.

The Graph Container Lemma. For every $c > 0$, there exists $\delta > 0$ such that the following holds. Let G be a graph with average degree d and maximum degree $\Delta(G) \le c \cdot d$, and set $\tau := 2\Delta/d$. There exists a collection $\mathcal{C}$ of subsets of $V(G)$, with

$$|\mathcal{C}| \le \binom{v(G)}{d\tau \cdot v(G)},$$

such that

(a) For every independent set $I \in I(G)$, there exists $C \in \mathcal{C}$ such that $I \subseteq C$.

(b) $|C| \le (1-\delta)v(G)$ for every $C \in \mathcal{C}$.

The Graph Container Algorithm. Given a graph G and an independent set $I \in I(G)$, we will maintain a partition $V(G) = A \cup S \cup X$, where A are the 'active' vertices, S is the current version of the fingerprint, and X is the set of 'excluded' vertices, which we already know are not in I. We start with $A = V(G)$ and $S = X = \emptyset$. Now, while $|X| \le \delta v(G)$, repeat the following steps:

1. Let v be the first vertex of I in the max-degree order on $G[A]$.
2. Move v into S, i.e., set $S := S \cup \{v\}$.
3. Move the neighbours of v into X, i.e., set $X := X \cup N(v)$.
4. Move the vertices which preceded v in the max-degree order on $G[A]$ into X, i.e., set

$$X := X \cup W, \quad \text{where } W = \{u \in A : u < v \text{ in the max-degree order on } G[A]\}.$$

5. Remove the new vertices of $S \cup X$ from A, i.e., set $A := V(G) \setminus (S \cup X)$.

Finally, set $A(I) := A$, $S(I) := S$, and $X(I) := X$.