Enumeration of intersection graph of x-monotone curves

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1 Basics

Definition 1 *Let $\mathcal{C}$ be a collection of sets. The intersection graph of $\mathcal{C}$ is the graph with vertex set $\mathcal{C}$ in which two vertices are adjacent if the two sets have nonempty intersection.*

Definition 2 *A string graph is an intersection graph of curves in the plane.*

Definition 3 *A collection of curves in the plane is called a collection of pseudosegments if each two have at most one point in common.*

Definition 4 *A curve in the plane is x-monotone if every vertical line intersects it exactly once.*

2 Lower bound

Theorem 5 *For $k \leq n^{1/3}$, there are at least $2^{\Omega(kn)}$ n-vertex intersection graphs of x-monotone pseudo-segments with clique number at most k.*

Corollary 6 *There are at least $2^{\Omega(n^{4/3})}$ labeled intersection graphs of x-monotone pseudo-segments.*

3 Interlude - VC dimension

Definition 7 *Let $\mathcal{F}$ be a set system. For a set X we define $\mathcal{F} \cap X = \{F \cap X : F \in \mathcal{F}\}$. We say that $\mathcal{F}$ shatters X if $\mathcal{F} \cap X = \mathcal{P}(X)$.*

Definition 8 *For a set-system $\mathcal{F}$, we define the VC-dimension of $\mathcal{F}$ to be the maximum integer k such that some set of size k is shattered by $\mathcal{F}$.*

Theorem 9 *Let $d \geq 2$ be fixed and $n, m \geq 2$. Then the number $h_d(n,m)$ of multiset systems of m subsets of $[n]$ with VC-dimension at most d satisfies*

$$h_d(m,n) = 2^{O(m^{1-1/d} n \log(m))}.$$

And if $m > n^d$, then

$$h_d(m,n) = 2^{O(n^d \log(m))}.$$

4 Necessary statements from other works - without proof

Definition 10 *A pseudoline is a two-way infinite x-monotone curve. An arrangement of pseudolines is a finite collection of pseudolines such that each two have at most one point in common. We say that two arrangements are x-isomorphic if a sweep by vertical line meets the crossings in the same order.*

Lemma 11 (Stanley, 1984) *The number of arrangements of m pseudolines, up to is x-isomorphisms, is at most $2^{\Theta(m^2 \log(m))}$.*

Lemma 12 (Bern et al., 1991) *Let $\mathcal{A}$ be a collection of pseudolines. Then, for any $\alpha \in \mathcal{A}$, the sum of the numbers of sides in all the cells in the arrangement of $\mathcal{A}$ that are supported by α is at most $O(m)$.*

Definition 13 *Let G be a graph. We define the VC-dimension of G to be the VC-dimension of the set system $\mathcal{F}(G) = \{N(v) : v \in G\}$. Where $N(v)$ is the neighbourhood of v.*

Lemma 14 (Pach and Tóth, 2006) *Let G be the intersection graph of a collection of pseudo-segments in the plane. Then the VC-dimension of G is at most an absolute constant d.*

Definition 15 *A collection of x-monotone pseudo-segments in the plane is* double grounded *if there are two lines l_1, l_2, such that each pseudo-segment in the collection has its left endpoint on l_1 and its right endpoint on l_2.*

Definition 16 *Let $\mathcal{A}$ be a collection of double grounded x-monotone pseudo-segments in the plane. The* vertical decomposition *of the arrangement of $\mathcal{A}$ is obtained by drawing a vertical segment from each crossing point and endpoint in the arrangement, in both directions, and extend it until it meets the arrangement of $\mathcal{A}$, else to $\pm\infty$. The cells of the arrangeet induced by the vertical decomposition are called* generalized trapezoids.

Lemma 17 (Clarckson and Shor, 1989) *Let $\mathcal{A}$ be a collection of m double grounded x-monotone pseudo-segments in the plane. Then for any parameter $1 \leq r \leq m$, there is a set of at most $s = 6r\log(m)$ curves in $\mathcal{A}$ whose vertical decomposition partitions the plane into t generalized trapezoids $\Delta_1, \Delta_2, \ldots, \Delta_t$, such that $t = O(s^2)$.*

5 Upper bound - main result

Let $f(m,n)$ denote the number of labelled intersection graphs between a collection $\mathcal{A}$ of m double grounded x-monotone curves whose grounds are the vertical lines at $x = 0$ and $x = 1$, and a collection $\mathcal{B}$ of n x-monotone curves whose endpoints lie inside the strip $S = [0,1] \times \mathbf{R}$ such that $A \cup B$ is a collection of pseudo-segments.

Lemma 18 *For $m, n \geq 1$, we have*

$$f(m,n) \leq 2^{O(n^{d(2d-1)} m^{(2d-2)/(2d-1)} \log^2(m)} + 2^{O(n^{3/2-1/d} \log(n)} + 2^{O(m \log^3(m)}$$

Theorem 19 *There is an absolute constant $\varepsilon \in (0,1)$ such that the following holds. There are at most $2^{O(n^{3/2-\varepsilon})}$ labelled n-vertex intersection graphs of x-monotone pseudo-segments in the plane.*

6 Some abbreviations I might use

- intersection graph → int graph
- x-monotone → x-mon
- isomorphism → iso
- collection → col
- VC-dimension → VC-dim
- endpoints → ends
- vertical → vert
- pseudo-segment → pseudo-seg
- decomposition → decomp
- between → btw