On Graph Classes with Minor Universal Elements: An Article by A. Georgakopoulos

Talk by Maximilian Strohmeier

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Let $<$ denote the graph minor relation. Unless otherwise specified, let morphisms to graphs be from minors, represented by functions mapping vertices to connected subgraphs and edges to paths. For a disjoint collection S of connected subgraphs, let $\langle S \rangle$ be the edge-maximal minor contracting each of the connected subgraphs. Minor-twin classes are equivalence classes of $<$.

Theorem 1.2. *The class of countable K_ω-minor-free graphs has no $<$-universal element.*

Proof. Suppose U is universal. Note that $U +_U v < U$. Let $f : U +_U v \to U$. Then $K_\omega \cong \left\langle f^{i+1}(v) \mid i \in \omega \right\rangle < U$. □

Corollary 1.3. *If, for every vertex v of a graph G, $G < G - v$ then there is no $<$-universal G-minor-free graph.*

Corollary. *There are uncountably many minor-twin classes of countable graphs.*

Proof. If there were countably many, the disjoint union of representatives for each not containing K_ω would be K_ω-minor-free $<$-universal. □

1 Uncountably many classes without universal elements

Theorem 3.2 (Matthiesen)**.** *There are uncountably many order-preserving $<$-equivalence classes of rooted binary trees.*

Proposition 3.1. *There are uncountably many classes of countable graphs without $<$-universal elements.*

2 Universal embeddable graphs

Let Σ be a closed orientable surface and Σ_G a closed orientable surface of minimal genus embedding graph G. Let morphisms from graphs to surfaces be embeddings. An embedding is generous if around every vertex exists an open disk only intersecting adjacent edges and around every edge exists an open disk only intersecting it.

Proposition 4.3. *Each countable graph embeddable in a surface is so generously.*

Proof. Adjust the embedding profinitely. □

Lemma 4.2. *Each countable graph embeddable in Σ is a minor of such a sub-cubic one.*

Proof. Pick a generous embedding and then replace vertices with binary trees in a surrounding disk. □

Theorem 4.4 (Youngs)**.** *The faces of finite G embedded into Σ_G are disks.*

Theorem 4.1. *The countable graphs componentwise embeddable in Σ have a $<$-universal element.*

Proof. The universal element is ω copies of all ways to "fill" faces of a $G \to \Sigma_G$ of all finite G embeddable to Σ. □

3 A universal K_5-free graph

Theorem 6.4 (Křìž and Thomas)**.** *Countable graphs are K_5-minor-free if and only if they are subgraphs of a tree-decomposable one with 3-simplex torsos pairwise intersecting on at most 3 vertices.*

Theorem 6.1. *There is a universal countable K_5-minor-free graph.*