Holes in Convex and Simple Drawings

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1 Definitions

Definition 1. A *simple drawing* of a graph is one where vertices are mapped to distinct points in the plane and edges are mapped to simple curves connecting the two corresponding vertices such that two edges have at most one point in common, which is either a common vertex or a proper crossing.

Definition 2. A triangle separates the plane into two connected components. The closure of each of the components is called a *side* of the triangle. A side S is *convex* if, for every pair of vertices in S, the connecting edge is fully contained in S. A simple drawing $\mathcal{D}$ of K_n is

- *convex* if every triangle in $\mathcal{D}$ has a convex side;
- *h-convex* (hereditarily convex) if there is a choice of a convex side S_T for every triangle T such that, for every triangle T' contained in S_T, it holds $S_{T'} \subseteq S_T$;
- *f-convex* (face convex) if there is a marking face F in the plane such that for all triangles the side not containing F is convex.

Definition 3. Consider a simple drawing of K_n. A *k-gon* $\mathcal{C}_k$ is a subdrawing isomorphic to the geometric drawing on k points in convex position. We call vertices in the interior of the convex side of $\mathcal{C}_k$ *interior* vertices. A *k-hole* is a k-gon that has no interior vertices.

2 Results

Theorem 1 (Empty Hexagon theorem for convex drawings)**.** *For every sufficiently large n, every convex drawing of K_n contains a 6-hole.*

Lemma 2. *Let $\mathcal{C}_k$ be a minimal k-gon in a convex drawing $\mathcal{D}$ of K_n with $n \geq k \geq 5$. Then the subdrawing $\mathcal{D}'$ induced by the vertices in the convex side of $\mathcal{C}_k$ is f-convex.*

Theorem 3. *Let $\mathcal{D}$ be a simple drawing of K_n with $n \geq 4$ and let v be a vertex of $\mathcal{D}$. Then $\mathcal{D}$ contains an empty 4-cycle passing through v.*

Corollary 4. *Every simple drawing of K_n with $n \geq 4$ contains at least $\frac{n}{4}$ empty 4-cycles.*