The matroid of a graphing an article by **Laszlo Lovasz**

Definition 1 (Local convergence).

$\mathcal{G}_D^\bullet := \{\textit{The space of all rooted connected degree D-bounded graphs}\} \quad, (\mathcal{G}_D^\bullet)_r := \{g \in \mathcal{G}_D^\bullet : \forall x \in V(g),\ d(x, root) \le r\}.$

A sequence of graphs $(G_1, G_2, \cdots)$ *is locally convergent, if for all* r *and* $g_r \in (\mathcal{G}_D^\bullet)_r$, $\mathbb{P}_x[B_r^{G_i}(x) \cong g_r]$ *converges.*

Definition 2 (Graphing). *A* ***graphing*** *is a quadruple* $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$*, where* $(J, \mathcal{B})$ *is a standard* ***Borel sigma-algebra****,* E *is a* ***Borel subset*** *of* $J \times J$ *that is symmetric (invariant under interchanging the coordinates),* λ *is a probability measure on* $(J, \mathcal{B})$*, and a* ***"measure-preservation"*** *condition holds:*

$$\int_A \deg_B(x)\, d\lambda(x) = \int_B \deg_A(x)\, d\lambda(x) \quad (A, B \in \mathcal{B}).$$

We also have this measure that can extend to any Borel set of $\mathcal{B} \times \mathcal{B}$*:*

$$\textit{Edge measure: } \eta(A \times B) = \int_A \deg_B(x)\, d\lambda(x)$$

A ***subgraphing***($\mathbf{G}^F$) *of a graphing* $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ *is a 4-tuple* $\mathbf{H} = (J, \mathcal{B}, F, \lambda)$*, where* $F \subseteq E$ *is a Borel set, we can prove this subgraphing is a graphing. and we have,*

$$\eta_{\mathbf{H}}(X) = \eta(X) \quad (X \in \mathcal{B} \times \mathcal{B}, X \subseteq F).$$

Lemma 3. *For* $B \in \mathcal{B}$*, the neighborhood* $N_{\mathbf{G}}(B)$ *is Borel.*

Lemma 4. *Let* $\mathbf{G}$ *be a graphing, and let* $f : V(\mathbf{G}) \times V(\mathbf{G}) \to \mathbb{R}$ *be a locally finite Borel function. Assume that* $f(x, y) = 0$ *unless* $y \in V(\mathbf{G}_x)$*. Then*

$$\int_{V(\mathbf{G})} \sum_y f(x, y)\, dx = \int_{V(\mathbf{G})} \sum_x f(x, y)\, dy.$$

Definition 5 (Matroid). *A* ***matroid*** $M = (E, \mathcal{I})$ *consists of a finite set* E *(the ground set) and a collection of subsets* $\mathcal{I}$ *(the independent sets), satisfying the following axioms:*

$$1) \underbrace{\emptyset \in \mathcal{I}}_{\textit{Non-emptiness}} \quad 2) \underbrace{A \in \mathcal{I},\ B \subseteq A \implies B \in \mathcal{I}}_{\textit{Hereditary}} \quad 3) \underbrace{A, B \in \mathcal{I},\ |A| > |B| \implies \exists x \in A \setminus B :\ B \cup \{x\} \in \mathcal{I}}_{\textit{Exchange property}}$$

or equivalently $M = (E, r)$ *consists of a finite set* E *(the ground set) and a rank function* $r : 2^E \to \mathbb{Z}_{\ge 0}$*, satisfying the following axioms:*

$$1) \underbrace{r(\emptyset) = 0}_{\textit{Normalization}} \quad 2) \underbrace{A \subseteq B \subseteq E : r(B) - |B \setminus A| \le r(A) \le r(B)}_{\textit{Monotonicity}} \quad 3) \underbrace{A, B \subseteq E : r(A \cup B) + r(A \cap B) \le r(A) + r(B)}_{\textit{Submodularity}}$$

Definition 6 (Cycle matroid of a graph).

$$G = (V, E) \to M(G) = (E, \mathcal{I} = \{\textit{All the cycle-free subsets of edges}\})$$

or equivalently

$$(E, r);\ r(X) := |V| - c;\ c : \textit{is the number of connected components}$$

Definition 7 ($\mathcal{B}$-partition). *Let* $(J, \mathcal{B})$ *be a standard Borel space, let* $\mathcal{P}$ *be a partition of* J*, and for* $X \subseteq J$*, define*

$$\mathcal{P}(X) = \cup\{P \in \mathcal{P} : P \cap X \ne \emptyset\}.$$

$\mathcal{P}$ *is a* $\mathcal{B}$*-partition, if* $A \in \mathcal{B} \implies \mathcal{P}(A) \in \mathcal{B}$.

Definition 8 (Re-randomization). *Let* π *be a probability measure on* $(J, \mathcal{B})$*, and let* $\mathbf{u}$ *be a random point from* π*. If* $\mathcal{P}_{\mathbf{u}}$ *is finite, then let* $\mathbf{v}$ *be a uniform random point of* P*; else, let* $\mathbf{v} = \mathbf{u}$*. We say that* $\mathcal{P}$ *has the re-randomizing property if* $\mathbf{v}$ *that is obtained by re-randomizing* $\mathbf{u}$ *along* $\mathcal{P}$ *distributed according to* π*.*

We will prove that the following rank normalization can be introduced as a cycle matroid for graphing. X_u is connected competent that includes u.

$$\rho(X) := \frac{r(X)}{|V|} = 1 - \mathrm{E}_{\mathbf{u}}\left(\frac{1}{|X_{\mathbf{u}}|}\right), \psi(X) := \mathrm{E}_{\mathbf{u}}\left(\frac{1}{|X_{\mathbf{u}}|}\right) \implies \rho_{\mathbf{G}}(X) := 1 - \mathrm{E}_{\mathbf{u}}\left(\frac{1}{|\mathbf{G}_{\mathbf{u}}^X|}\right) = 1 - \psi(\mathcal{P}^X).$$

Lemma 9. *Let $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ be a graphing, and let $\mathcal{P}$ be the partition of J into the connected components of $\mathbf{G}$. Then $\mathcal{P}$ is a $\mathcal{B}$-partition with the re-randomizing property.*

Lemma 10. *(a) If P and Q are $\mathcal{B}$-partitions, then so is $P \vee Q$.*
(b) If Q has a countable number of classes, then $P \wedge Q$ is a $\mathcal{B}$-partition.

Lemma 11. *Let $\mathcal{P}$ be a $\mathcal{B}$-partition, $A \in \mathcal{B}$, and let $A_k = \cup\{P \in \mathcal{P} : |P \cap A| = k\}$. Then $A_k \in \mathcal{B}$ for $k = 0, 1, \ldots, \infty$.*

Definition 12 (finite-class representative)**.** *Call a set S a finite-class representative of a $\mathcal{B}$-partition $\mathcal{P}$, if it is a Borel set, $|S \cap P| = 1$ for every finite $P \in \mathcal{P}$, and $|S \cap P| = 0$ for every infinite $P \in \mathcal{P}$.*

Lemma 13. *Every $\mathcal{B}$-partition has a finite-class representative.*

Lemma 14. *If $\mathcal{P}'$ is obtained by splitting some finite classes of $\mathcal{P}$, and $\mathcal{P}$ has the re-randomizing property, then so does $\mathcal{P}'$.*

Lemma 15. *If $\mathcal{P}$and $\mathcal{Q}$ are $\mathcal{B}$-partitions with the re-randomizing property, then so does $\mathcal{P} \vee \mathcal{Q}$.*

Lemma 16 (Main lemma)**.** *For $\mathcal{B}$-partitions $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R} \leq \mathcal{P} \wedge \mathcal{Q}$ with the re-randomizing property,*

$$\psi(\mathcal{R}) + \psi(\mathcal{P} \vee \mathcal{Q}) \geq \psi(\mathcal{P}) + \psi(\mathcal{Q}),$$

Theorem 17 (Submodularity of the normalized rank function of a graphing)**.** *The set function ρ, defined on the Borel subsets of the edge set of a graphing, is increasing and submodular. Additionally, it satisfies the inequalities:*

$$\frac{1}{1+D}\eta(X) \leq \rho(X) \leq \eta(X),$$

where D is the maximum degree of the graphing.

Definition 18. ***matroid polytope****,is the convex hull of indicator vectors of subforests. The points of this polytope are called* ***fractional independent sets*** *for the cycle matroid of a graphing those are* ***minorizing measures*** *φ with $\varphi(\emptyset) = 0$ and $0 \leq \alpha \leq \varphi$. It is also a* ***maximal minorizing measure*** *if $\alpha(E) = \varphi(E)$. A minorizing measure is* ***extremal****, if it cannot be written as the average of two different minorizing measures.*

Definition 19 (Hyperfinite graphing)**.** *We say that a graphing $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$ is hyperfinite, if for every $\varepsilon > 0$ there is a Borel set $X \subseteq E$ with $\eta(X) \leq \varepsilon$ such that all connected components of $\mathbf{G} \backslash X$ are finite.*

Lemma 20. *Let $\mathbf{F} = (J, \mathcal{B}, E, \lambda)$ be a hyperfinite acyclic graphing. Then for all Borel sets $U \subseteq E$, we have $\rho(U) = \eta(U)/2$.*

In particular, ρ is not only submodular, but modular, and hence, a measure.

Theorem 21. *Let $\mathbf{G}$ be a hyperfinite graphing and $\mathbf{H} = (J, F)$, a Borel measurable spanning subforest of $\mathbf{G}$. Then*

$$\alpha(X) = \frac{1}{2}\eta_{\mathbf{G}}(X \cap F)$$

defines an extremal maximal minorizing measure on the Borel subsets of E.

Theorem 22. *If $G_1, G_2, \ldots$ is a sequence of finite graphs with all degrees bounded by $D \geq 0$ locally converging to a graphing $\mathbf{G} = (J, \mathcal{B}, E, \lambda)$, then $\bar{\rho}(G_n) \to \bar{\rho}(\mathbf{G})$ as $n \to \infty$. Which $\bar{\rho}(\mathbf{G}) := \rho_{\mathbf{G}}(E(\mathbf{G}))$, is total rank.*