The Quantitative Fractional Helly Theorem

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24 October 2024

1 Introduction

Theorem 1 (Helly's Theorem, 1913/1921) *Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$ such that any $d+1$ elements $K_1, ..., K_{d+1}$ of $\mathcal{F}$ intersect: $\bigcap_{i=1}^{d+1} K_i \neq \emptyset$. Then $\bigcap \mathcal{F} \neq \emptyset$, that is, all sets in $\mathcal{F}$ intersect.*

The number $d+1$ in the Theorem is optimal, but these strengthenings exist:

Theorem 2 (Colorful Helly Theorem, Lovász '82) *Let $\mathcal{F}_1, ..., \mathcal{F}_{d+1}$ be families of convex sets in $\mathbb{R}^d$ such that every transversal $K_1 \in \mathcal{F}_1, ..., K_{d+1} \in \mathcal{F}_{d+1}$ intersects. Then there is $i \in [d+1]$ such that $\bigcap \mathcal{F}_i \neq \emptyset$.*

Theorem 3 (Quantitative Helly Theorem, Bárány, Katchalski, Pach '82) *There exists a constant $c > 0$ such that for any family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ such that any $2d$ elements $K_1, ..., K_{2d}$ of $\mathcal{F}$ intersect in a set of volume at least 1, i.e.*

$$\mathrm{Vol}\left(\bigcap_{i=1}^{2d} K_i\right) \geq 1 \Rightarrow \mathrm{Vol}\left(\bigcap \mathcal{F}\right) \geq v(d) > 0.$$

That means all sets intersect in a set of volume at least $v(d)$.

Again, the number $2d$ cannot be reduced and the best possible function $v(d)$ is known to be between $(cd)^{-\frac{d}{2}}$ and $(cd)^{-\frac{3d}{2}}$ for some constant c. (Naszódi '16; Brazitikos '17).

Theorem 4 (Fractional Helly Theorem, Katchalski, Liu '79) *Let $\mathcal{F}$ be a family of n convex sets in $\mathbb{R}^d$ and $\alpha > 0$ such that at least $\alpha\binom{n}{d+1}$ choices of $d+1$ elements $K_1, ..., K_{d+1}$ of $\mathcal{F}$ intersect, i.e. $\bigcap_{i=1}^{d+1} K_i \neq \emptyset$*

Then there exists $\beta = \beta(d, \alpha)$, such that at least βn sets of $\mathcal{F}$ intersect, that is $m \geq \beta n$ elements $K_1, ..., K_m$ of $\mathcal{F}$ exist such that $\bigcap_{i=1}^{m} K_i \neq \emptyset$.

The best function for β is $\beta = 1 - (1-\alpha)^{\frac{1}{d+1}}$ (Kalai '84).

Theorem 5 (Quantitative Fractional Helly Theorem) *For all $d \in \mathbb{N}$, there exists $v(d) > 0$, and for every $\alpha \in (0,1)$, there exists $\beta(d,\alpha) > 0$, such that the following holds:*
Let $\mathcal{F}$ be a family of n convex sets in $\mathbb{R}^d$ such that at least $\alpha\binom{n}{d+1}$ of the $(d+1)$-tuples of members of $\mathcal{F}$ have an intersection of volume at least 1. Then one can find at least βn sets in $\mathcal{F}$ whose intersection has volume at least $v(d)$.

This is what we will prove. The values of β and v are not optimized but the proof gives $v(d) = 2^{-2^{2^{\cdot^{\cdot^{\cdot}}}}}$, where the tower is of height $\mathcal{O}(d)$.

2 Prerequisites

The first published proof of Helly's Theorem is due to Radon and uses this fact:

Theorem 6 (Radon's Theorem, 1921) *Any collection of $d{+}2$ vectors $x_1, ..., x_{d+2}$ in $\mathbb{R}^d$ admits a partition $X_1 \cup X_2$ into two parts, whose convex hulls intersect:* $\operatorname{conv}(x_i : x_i \in X_1) \cap \operatorname{conv}(x_i : x_i \in X_2) \neq \emptyset$.

The set of convex compact sets of non-empty interior in $\mathbb{R}^d$ is also called set of convex bodies $\mathcal{K}(\mathbb{R}^d)$. We can give it the structure of a metric space using the following definition:

Definition 1 *The* Hausdorff distance *of two sets X and Y in a metric space (M,d) is defined as $d_H(X,Y) = \max(\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X))$, where $d(x,Y) := \inf_{y\in Y} d(x,y)$.*

The set of compact sets in some metric space (M,d) equipped with this metric becomes a metric space itself. This space is compact if M is compact. We will also need the following containment theorems of Convex Geometry:

Theorem 7 (John's Ellipsoid Theorem, '48) *Any convex set $K \subset \mathbb{R}^d$ admits a unique ellipsoid E of minimal volume containing K and if E is centered at 0, then $\frac{1}{d}E \subset K$, that is shrinking the ellipsoid by a factor of $\frac{1}{d}$ towards its center gives an ellipsoid contained in K.*

Theorem 8 (Simplex containment Theorem, Galicer, Merzbacher, Pinasco '19) *Let K be a convex body in $\mathbb{R}^d$. Then there exists a simplex of volume $\mathcal{O}(d)^{\frac{d}{2}} \cdot \operatorname{Vol}(K)$ containing K.*

Lemma 1 *N hyperplanes cut $\mathbb{R}^d$ into at most $\sum_{k=0}^{d}\binom{N}{k} \leq N^d$ cells.*

Further we need some results from extremal hypergraph theory:

Theorem 9 (h-partite Ramsey theorem, follows from Erdős '64) *For all $h, m, q \in \mathbb{N}, \exists M \in \mathbb{N}$: Every q-coloring of the edges of the complete h-partite h-uniform hypergraph $L_h(M)$ contains a monochromatic copy of $L_h(m)$.*

Lemma 2 (Hypergraph Saturation, Erdős, Simonovits '83) *For all $h, m \in \mathbb{N}, \alpha > 0 \exists \gamma > 0$: Every h-uniform hypergraph on a sufficient large number n of vertices with at least $\alpha\binom{n}{h}$ edges contains at least γn^{hm} copies of $L_h(m)$.*